

# Learning the Structure and Parameters of Large-Population Graphical Games from Behavioral Data

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## Abstract

We formalize and study the problem of learning the structure and parameters of *graphical games* from strictly *behavioral* data. We cast the problem as a maximum likelihood estimation based on a generative model defined by the *pure-strategy Nash equilibria* of the game. The formulation brings out the interplay between goodness-of-fit and model complexity: good models capture the equilibrium behavior represented in the data while controlling the *true* number of equilibria, including those potentially unobserved. We provide a generalization bound for maximum likelihood estimation. We discuss several optimization algorithms including *convex loss minimization*, sigmoidal approximations and exhaustive search. We formally prove that games in our hypothesis space have a small *true* number of equilibria, with high probability; thus, convex loss minimization is sound. We illustrate our approach, show and discuss promising results on synthetic data and the U.S. congressional voting records.

## 1 Introduction

*Graphical games* [Kearns et al., 2001] were one of the first and most influential graphical models for game theory. It has been about a decade since their introduction to the AI community. There has also been considerable progress on problems of *computing* classical equilibrium solution concepts such as Nash [Nash, 1951] and correlated equilibria [Aumann, 1974] in graphical games (see, e.g., Kearns et al. [2001], Vickrey and Koller [2002], Ortiz and Kearns [2002], Blum et al. [2006], Kakade et al. [2003], Papadimitriou and Roughgarden [2008], Jiang and Leyton-Brown [2011] and the references therein). Indeed, graphical games played a prominent role in establishing the computational complexity of computing Nash equilibria in general normal-form games (see, e.g., Daskalakis et al. [2009] and the references therein).

Relatively less attention has been paid to the problem of *learning* the structure of graphical games from data. Addressing this problem is essential to the development, potential use and success of game-theoretic models in practical applications.

Indeed, we are beginning to see an increase in the availability of data collected from processes that are the result of deliberate actions of agents in complex system. A lot of this

data results from the interaction of a large number of individuals, being people, companies, governments, groups or engineered autonomous systems (e.g. autonomous trading agents), for which any form of global control is usually weak. The Internet is currently a major source of such data, and the smart grid, with its trumpeted ability to allow individual customers to install autonomous control devices and systems for electricity demand, will likely be another one in the near future.

We present a formal framework and design algorithms for learning the structure and parameters of graphical games [Kearns et al., 2001] in large populations of agents. We concentrate on learning from purely behavioral data. We expect that, in most cases, the parameters quantifying a utility function or best-response condition are unavailable and hard to determine in real-world settings. The availability of data resulting from the observation of an individual *public behavior* is arguably a weaker assumption than the availability of individual *utility* observations, which are often *private*.

Our technical contributions include a novel generative model of behavioral data in Section 4 for general games. We define identifiability and triviality of games. We provide conditions which ensures identifiability among non-trivial games. We then present the maximum likelihood problem for general (non-trivial identifiable) games. In Section 5, we show a generalization bound for the maximum likelihood problem as well as an upper bound of the VC-dimension of influence games. In Section 6, we approximate the original problem by maximizing the number of observed equilibria in the data, suitable for a hypothesis space of games with small *true* number of equilibria. We then present our convex loss minimization approach and a baseline sigmoidal approximation for (linear) influence games. We also present exhaustive search methods for both general as well as influence games. In Section 7, we define absolute-indifference of players and show that our convex loss minimization approach produces games in which all players are non-absolutely-indifferent. We provide a distribution-free bound which shows that linear influence games have small *true* number of equilibria with high probability.

## 2 Related Work

Our work *complements* the recent line of work on learning graphical games [Vorobeychik et al., 2005, Ficici et al., 2008, Duong et al., 2009, Gao and Pfeffer, 2010, Ziebart et al., 2010, Waugh et al., 2011]. With the exception of Ziebart et al. [2010], Waugh et al. [2011], previous methods assume that the actions as well as corresponding payoffs (or noisy samples from the true payoff function) are observed in the data. Another notable exception is a recently proposed framework from the learning theory community to model *collective* behavior [Kearns and Wortman, 2008]. The approach taken there considers dynamics and is based on stochastic models. Our work differs from methods that assume that the game is known [Wright and Leyton-Brown, 2010]. The work of Vorobeychik et al. [2005], Gao and Pfeffer [2010], Wright and Leyton-Brown [2010], Ziebart et al. [2010] present experimental validation mostly for 2 players only, 7 players in Waugh et al. [2011] and up to 13 players in Duong et al. [2009].

In this paper, we assume that the joint-actions is the only observable information. To the best of our knowledge, we present the first techniques for learning the structure and parameters of large-population graphical games from joint-actions only. Furthermore, we

present experimental validation in games of up to 100 players. Our convex loss minimization approach could potentially be applied to larger problems since it is polynomial-time.

There has been a significant amount of work for learning the structure of *probabilistic* graphical models from data. We mention only a few references that follow a maximum likelihood approach for Markov random fields [Lee et al., 2006], bounded tree-width distributions [Chow and Liu, 1968, Srebro, 2001], Ising models [Wainwright et al., 2006, Banerjee et al., 2008, Höfling and Tibshirani, 2009], Gaussian graphical models [Banerjee et al., 2006], Bayesian networks [Guo and Schuurmans, 2006, Schmidt et al., 2007b] and directed cyclic graphs [Schmidt and Murphy, 2009].

Our approach learns the structure and parameters of games by maximum likelihood estimation on a related probabilistic model. Our probabilistic model does not fit into any of the types described above. Although a (directed) graphical game has a directed cyclic graph, there is a semantic difference with respect to graphical models. Structure in a graphical model implies a factorization of the probabilistic model. In a graphical game, the graph structure implies *strategic* dependence between players, and has no immediate probabilistic implication. Furthermore, our general model differs from Schmidt and Murphy [2009] since our generative model does not decompose as a multiplication of potential functions.

### 3 Background

In classical game-theory (see, e.g. Fudenberg and Tirole [1991] for a textbook introduction), a *normal-form game* is defined by a set of *players*  $V$  (e.g. we can let  $V = \{1, \dots, n\}$  if there are  $n$  players), and for each player  $i$ , a set of *actions*, or *pure-strategies*  $A_i$ , and a payoff function  $u_i : \times_{j \in V} A_j \rightarrow \mathbb{R}$  mapping the joint-actions of all the players, given by the Cartesian product  $\mathcal{A} \equiv \times_{j \in V} A_j$ , to a real number. In non-cooperative game theory we assume players are greedy, rational and act independently, by which we mean that each player  $i$  always want to maximize their own utility, subject to the actions selected by others, irrespective of how the optimal action chosen help or hurt others.

A core solution concept in non-cooperative game theory is that of an *Nash equilibrium*. A joint-action  $\mathbf{x}^* \in \mathcal{A}$  is a *pure-strategy Nash equilibrium* of a non-cooperative game if, for each player  $i$ ,  $x_i^* \in \arg \max_{x_i \in A_i} u_i(x_i, \mathbf{x}_{-i}^*)$ ; that is,  $\mathbf{x}^*$  constitutes a *mutual best-response*, no player  $i$  has any incentive to unilaterally deviate from the prescribed action  $x_i^*$ , given the joint-action of the other players  $\mathbf{x}_{-i}^* \in \times_{j \in V - \{i\}} A_j$  in the equilibrium.

In what follows, we denote a game by  $\mathcal{G}$ , and the set of all *pure-strategy Nash equilibria* of  $\mathcal{G}$  by:

$$\mathcal{NE}(\mathcal{G}) \equiv \{\mathbf{x}^* \mid (\forall i \in V) x_i^* \in \arg \max_{x_i \in A_i} u_i(x_i, \mathbf{x}_{-i}^*)\} \quad (1)$$

A (*directed*) *graphical game* is a game-theoretic graphical model [Kearns et al., 2001]. It provides a succinct representation of normal-form games. In a graphical game, we have a (directed) graph  $G = (V, E)$  in which each node in  $V$  corresponds to a player in the game. The interpretation of the edges/arcs  $E$  of  $G$  is that the payoff function of player  $i$  is only a function of the set of parents/neighbors  $\mathcal{N}_i \equiv \{j \mid (i, j) \in E\}$  in  $G$  (i.e. the set of players corresponding to nodes that point to the node corresponding to player  $i$  in the graph). In the context of a graphical game, we refer to the  $u_i$ 's as the *local payoff functions/matrices*.

*Linear influence games* [Irfan and Ortiz, 2011] are a sub-class of graphical games. For linear influence games, we assume that we are given a matrix of influence weights  $\mathbf{W} \in \mathbb{R}^{n \times n}$ , with  $\text{diag}(\mathbf{W}) = \mathbf{0}$ , and a threshold vector  $\mathbf{b} \in \mathbb{R}^n$ . For each player  $i$ , we define the influence function  $f_i(\mathbf{x}_{-i}) \equiv \sum_{j \in \mathcal{N}_i} w_{ij} x_j - b_i = \mathbf{w}_{i,-i}^T \mathbf{x}_{-i} - b_i$  and the payoff function  $u_i(\mathbf{x}) \equiv x_i f_i(\mathbf{x}_{-i})$ . We further assume binary actions:  $A_i \equiv \{-1, +1\}$  for all  $i$ . The *best response*  $x_i^*$  of player  $i$  to the joint-action  $\mathbf{x}_{-i}$  of the other players is defined as:

$$\left\{ \begin{array}{l} \mathbf{w}_{i,-i}^T \mathbf{x}_{-i} > b_i \Rightarrow x_i^* = +1, \\ \mathbf{w}_{i,-i}^T \mathbf{x}_{-i} < b_i \Rightarrow x_i^* = -1 \text{ and} \\ \mathbf{w}_{i,-i}^T \mathbf{x}_{-i} = b_i \Rightarrow x_i^* \in \{-1, +1\} \end{array} \right\} \Leftrightarrow x_i^* (\mathbf{w}_{i,-i}^T \mathbf{x}_{-i} - b_i) \geq 0 \quad (2)$$

Hence, for any other player  $j$ ,  $w_{ij} \in \mathbb{R}$  can be thought as a *weight* parameter quantifying the “influence factor” that  $j$  has on  $i$ , and  $b_i \in \mathbb{R}$  as a *threshold* parameter to the level of “tolerance” that player  $i$  has for playing  $-1$ .

As discussed in Irfan and Ortiz [2011], linear influence games are also a sub-class of poly-matrix games [Janovskaja, 1968]. Furthermore, in the special case of  $\mathbf{b} = \mathbf{0}$  and symmetric  $\mathbf{W}$ , a linear influence game becomes a *party-affiliation game* [Fabrikant et al., 2004].

Figure 3 provides a preview illustration of the application of our approach to congressional voting.

## 4 Preliminaries

Our goal is to learn the structure and parameters of a graphical game from observed joint-actions. Note that our problem is unsupervised, i.e. we do not know a priori which joint-actions are equilibria and which ones are not. If our only goal were to find a game  $\mathcal{G}$  in which all the given observed data is an equilibrium, then a “dummy” influence game with  $\mathcal{G} = (\mathbf{W}, \mathbf{b})$ ,  $\mathbf{W} = \mathbf{0}$ ,  $\mathbf{b} = \mathbf{0}$  would be the optimal solution since  $|\mathcal{NE}(\mathcal{G})| = 2^n$ . In this section, we present a probabilistic formulation that allows finding games that maximize the *empirical proportion of equilibria* in the data while keeping the *true proportion of equilibria* as low as possible. Furthermore, we show that *trivial* games such as  $\mathbf{W} = \mathbf{0}$ ,  $\mathbf{b} = \mathbf{0}$ , obtain the lowest log-likelihood.

### 4.1 On the Identifiability of Games

Several games with different coefficients can lead to the same Nash equilibria set. As a simple example that illustrates the issue of identifiability, consider the three following influence games with the same Nash equilibria sets, i.e.  $\mathcal{NE}(\mathbf{W}_k, \mathbf{0}) = \{(-1, -1, -1), (+1, +1, +1)\}$  for  $k = 1, 2, 3$ :

$$\mathbf{W}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1/2 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad \mathbf{W}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathbf{W}_3 = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Clearly, using structural properties alone, one would generally prefer the former two models to the latter, all else being equal (e.g. generalization performance). A large number of the econometrics literature concerns the issue of identifiability of models from data. In

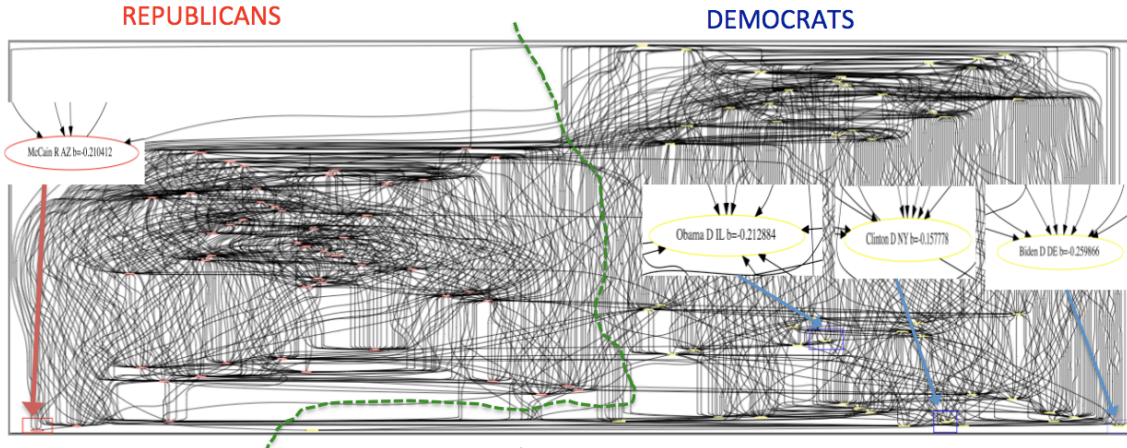


Figure 1: **110th US Congress’s Linear Influence Game (January 3, 2007-09)**: We provide an illustration of the application of our approach to real congressional voting data. Irfan and Ortiz [2011] use such LIGs to address a variety of computational problems, including the identification of *most influential* senators. We show the graph connectivity of a LIG learnt by independent  $\ell_1$ -regularized logistic regression (see Sect. 6.5). We highlight some characteristics of the graph, consistent with anecdotal evidence. First, senators are more likely to be influenced by members of the same party than by members of the opposite party (the dashed green line denotes the separation between the parties). Republicans were “more strongly united” (tighter connectivity) than Democrats at the time. Second, the current US Vice President Biden (Dem./Delaware) and McCain (Rep./Arizona) are displayed at the “extreme of each party” (Biden at the bottom-right corner, McCain at the bottom-left) eliciting their opposite ideologies. Third, note that Biden, McCain, the current US President Obama (Dem./Illinois) and US Secretary of State Hillary Clinton (Dem./New York) have very few outgoing arcs; e.g., Obama only directly influences Feingold (Dem./Wisconsin), a prominent senior member with strongly liberal stands. One may wonder why do such prominent senators seem to have so little direct influence on others? A possible explanation is that US President Bush was about to complete his second term (the maximum allowed). Both parties had *very long* presidential primaries. All those senators contended for the presidential candidacy within their parties. Hence, one may posit that those senators were focusing on running their campaigns and that their influence in the *day-to-day* business of congress was channeled through other prominent senior members of their parties.

typical machine-learning fashion, we side-step this issue by measuring the quality of our data-induced models via their generalization ability and invoke the principle of Ockham’s razor to bias our search toward simpler models using well-known and -studied regularization techniques. In particular, we take the view that games are identifiable by their Nash equilibria. Hence our next definition.

**Definition 1.** We say that two games  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are equivalent if and only if their Nash equilibria sets are identical, i.e.:  $\mathcal{G}_1 \equiv_{\mathcal{NE}} \mathcal{G}_2 \Leftrightarrow \mathcal{NE}(\mathcal{G}_1) = \mathcal{NE}(\mathcal{G}_2)$ .

## 4.2 Generative Model of Behavioral Data

We propose the following generative model for behavioral data based strictly in the context of “simultaneous”/one-shot play in non-cooperative game theory. Let  $\mathcal{G}$  be a game. With some probability  $0 < q < 1$ , a joint-action  $\mathbf{x}$  is chosen uniformly at random from  $\mathcal{NE}(\mathcal{G})$ ; otherwise,  $\mathbf{x}$  is chosen uniformly at random from its complement set  $\{-1, +1\}^n - \mathcal{NE}(\mathcal{G})$ . Hence, the generative model is a mixture model with mixture parameter  $q$  corresponding

to the probability that a stable outcome (i.e. a Nash equilibrium) of the game is observed. Formally, the probability mass function (PMF) over joint-behaviors  $\{-1, +1\}^n$  parametrized by  $(\mathcal{G}, q)$  is:

$$p_{(\mathcal{G}, q)}(\mathbf{x}) = q \frac{1[\mathbf{x} \in \mathcal{NE}(\mathcal{G})]}{|\mathcal{NE}(\mathcal{G})|} + (1 - q) \frac{1[\mathbf{x} \notin \mathcal{NE}(\mathcal{G})]}{2^n - |\mathcal{NE}(\mathcal{G})|} \quad (3)$$

where we can think of  $q$  as the “signal” level, and thus  $1 - q$  as the “noise” level in the data set.

**Remark 2.** Note that in order for eq.(3) to be a valid PMF for any  $\mathcal{G}$ , we need to enforce the following conditions  $|\mathcal{NE}(\mathcal{G})| = 0 \Rightarrow q = 0$  and  $|\mathcal{NE}(\mathcal{G})| = 2^n \Rightarrow q = 1$ . Furthermore, note that in both cases ( $|\mathcal{NE}(\mathcal{G})| \in \{0, 2^n\}$ ) the PMF becomes a uniform distribution. On the other hand, if  $0 < |\mathcal{NE}(\mathcal{G})| < 2^n$  then setting  $q \in \{0, 1\}$  leads to an invalid PMF.

Let  $\pi(\mathcal{G})$  be the true proportion of equilibria in the game  $\mathcal{G}$  relative to all possible joint-actions, i.e.:

$$\pi(\mathcal{G}) \equiv |\mathcal{NE}(\mathcal{G})|/2^n \quad (4)$$

**Definition 3.** We say that a game  $\mathcal{G}$  is trivial if and only if  $|\mathcal{NE}(\mathcal{G})| \in \{0, 2^n\}$  (or equivalently  $\pi(\mathcal{G}) \in \{0, 1\}$ ), and non-trivial if and only if  $0 < |\mathcal{NE}(\mathcal{G})| < 2^n$  (or equivalently  $0 < \pi(\mathcal{G}) < 1$ ).

The following propositions establish that the condition  $q > \pi(\mathcal{G})$  ensures that the probability of an equilibrium is strictly greater than a non-equilibrium. The condition also guarantees identifiability among non-trivial games.

**Proposition 4.** Given a non-trivial game  $\mathcal{G}$ , the mixture parameter  $q > \pi(\mathcal{G})$  if and only if  $p_{(\mathcal{G}, q)}(\mathbf{x}_1) > p_{(\mathcal{G}, q)}(\mathbf{x}_2)$  for any  $\mathbf{x}_1 \in \mathcal{NE}(\mathcal{G})$  and  $\mathbf{x}_2 \notin \mathcal{NE}(\mathcal{G})$ .

*Proof.* Note that  $p_{(\mathcal{G}, q)}(\mathbf{x}_1) = q/|\mathcal{NE}(\mathcal{G})| > p_{(\mathcal{G}, q)}(\mathbf{x}_2) = (1 - q)/(2^n - |\mathcal{NE}(\mathcal{G})|) \Leftrightarrow q > |\mathcal{NE}(\mathcal{G})|/2^n$  and given eq.(4), we prove our claim.  $\square$

**Proposition 5.** Let  $\mathcal{G}_1$  and  $\mathcal{G}_2$  be two non-trivial games. For some mixture parameter  $q > \max(\pi(\mathcal{G}_1), \pi(\mathcal{G}_2))$ ,  $\mathcal{G}_1$  and  $\mathcal{G}_2$  are equivalent if and only if they induce the same PMF over the joint-action space  $\{-1, +1\}^n$  of the players, i.e.:  $\mathcal{G}_1 \equiv_{\mathcal{NE}} \mathcal{G}_2 \Leftrightarrow (\forall \mathbf{x}) p_{(\mathcal{G}_1, q)}(\mathbf{x}) = p_{(\mathcal{G}_2, q)}(\mathbf{x})$ .

*Proof.* Let  $\mathcal{NE}_k \equiv \mathcal{NE}(\mathcal{G}_k)$ . First, we prove the  $\Rightarrow$  direction. By Definition 1,  $\mathcal{G}_1 \equiv_{\mathcal{NE}} \mathcal{G}_2 \Rightarrow \mathcal{NE}_1 = \mathcal{NE}_2$ . Note that  $p_{(\mathcal{G}_k, q)}(\mathbf{x})$  in eq.(3) depends only on characteristic functions  $1[\mathbf{x} \in \mathcal{NE}_k]$ . Therefore,  $(\forall \mathbf{x}) p_{(\mathcal{G}_1, q)}(\mathbf{x}) = p_{(\mathcal{G}_2, q)}(\mathbf{x})$ .

Second, we prove the  $\Leftarrow$  direction by contradiction. Assume  $(\exists \mathbf{x}) \mathbf{x} \in \mathcal{NE}_1 \wedge \mathbf{x} \notin \mathcal{NE}_2$ .  $p_{(\mathcal{G}_1, q)}(\mathbf{x}) = p_{(\mathcal{G}_2, q)}(\mathbf{x})$  implies that  $q/|\mathcal{NE}_1| = (1 - q)/(2^n - |\mathcal{NE}_2|) \Rightarrow q = |\mathcal{NE}_1|/(2^n + |\mathcal{NE}_1| - |\mathcal{NE}_2|)$ . Since  $q > \max(\pi(\mathcal{G}_1), \pi(\mathcal{G}_2)) \Rightarrow q > \max(|\mathcal{NE}_1|, |\mathcal{NE}_2|)/2^n$  by eq.(4). Therefore  $\max(|\mathcal{NE}_1|, |\mathcal{NE}_2|)/2^n < |\mathcal{NE}_1|/(2^n + |\mathcal{NE}_1| - |\mathcal{NE}_2|)$ . If we assume that  $|\mathcal{NE}_1| \geq |\mathcal{NE}_2|$  we reach the contradiction  $|\mathcal{NE}_1| - |\mathcal{NE}_2| < 0$ . If we assume that  $|\mathcal{NE}_1| \leq |\mathcal{NE}_2|$  we reach the contradiction  $(2^n - |\mathcal{NE}_2|)(|\mathcal{NE}_2| - |\mathcal{NE}_1|) < 0$ .  $\square$

**Remark 6.** Recall that a trivial game induces a uniform PMF by Remark 2. Therefore, a non-trivial game is not equivalent to a trivial game since by Proposition 4, non-trivial games do not induce uniform PMFs.

### 4.3 Learning the Structure of Games via Maximum Likelihood Estimation

The *learning problem* consists on estimating the structure and parameters of a graphical game from data. We point out that our problem is unsupervised, i.e. we do not know a priori which joint-actions are equilibria and which ones are not. We based our framework on the fact that games are identifiable with respect to their induced PMF by Proposition 5.

First, we introduce a shorthand notation for the Kullback-Leibler (KL) divergence between two Bernoulli distributions parametrized by  $0 \leq p_1 \leq 1$  and  $0 \leq p_2 \leq 1$ :

$$\begin{aligned} KL(p_1 \| p_2) &\equiv KL(\text{Bernoulli}(p_1) \| \text{Bernoulli}(p_2)) \\ &= p_1 \log \frac{p_1}{p_2} + (1 - p_1) \log \frac{1-p_1}{1-p_2} \end{aligned} \quad (5)$$

Using this function, we can derive the following expression of the maximum likelihood estimation problem.

**Lemma 7.** *Given a dataset  $\mathcal{D} = \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$ , let  $\hat{\pi}(\mathcal{G})$  be the empirical proportion of equilibria, i.e. the proportion of samples in  $\mathcal{D}$  that are equilibria of  $\mathcal{G}$ :*

$$\hat{\pi}(\mathcal{G}) \equiv \frac{1}{m} \sum_l 1[\mathbf{x}^{(l)} \in \mathcal{NE}(\mathcal{G})] \quad (6)$$

*the maximum likelihood estimation problem for the probabilistic model in eq.(3) can be expressed as:*

$$\max_{(\mathcal{G}, q) \in \Upsilon} \hat{\mathcal{L}}(\mathcal{G}, q) \quad , \quad \hat{\mathcal{L}}(\mathcal{G}, q) = KL(\hat{\pi}(\mathcal{G}) \| \pi(\mathcal{G})) - KL(\hat{\pi}(\mathcal{G}) \| q) - n \log 2 \quad (7)$$

where  $\mathcal{H}$  is the class of games of interest,  $\Upsilon = \{(\mathcal{G}, q) \mid \mathcal{G} \in \mathcal{H} \wedge 0 < \pi(\mathcal{G}) < q < 1\}$  is the hypothesis space of non-trivial identifiable games,  $\pi(\mathcal{G})$  is defined as in eq.(4) and the optimal mixture parameter  $\hat{q} = \min(\hat{\pi}(\mathcal{G}), 1 - \frac{1}{2m})$ .

*Proof.* Let  $\mathcal{NE} \equiv \mathcal{NE}(\mathcal{G})$ ,  $\pi \equiv \pi(\mathcal{G})$  and  $\hat{\pi} \equiv \hat{\pi}(\mathcal{G})$ . First, for a non-trivial  $\mathcal{G}$ ,  $\log p_{(\mathcal{G}, q)}(\mathbf{x}^{(l)}) = \log \frac{q}{|\mathcal{NE}|}$  for  $\mathbf{x}^{(l)} \in \mathcal{NE}$ , and  $\log p_{(\mathcal{G}, q)}(\mathbf{x}^{(l)}) = \log \frac{1-q}{2^n - |\mathcal{NE}|}$  for  $\mathbf{x}^{(l)} \notin \mathcal{NE}$ . The average log-likelihood  $\hat{\mathcal{L}}(\mathcal{G}, q) = \frac{1}{m} \sum_l \log p_{\mathcal{G}, q}(\mathbf{x}^{(l)}) = \hat{\pi} \log \frac{q}{|\mathcal{NE}|} + (1 - \hat{\pi}) \log \frac{1-q}{2^n - |\mathcal{NE}|} = \hat{\pi} \log \frac{q}{\pi} + (1 - \hat{\pi}) \log \frac{1-q}{1-\pi} - n \log 2$ . By adding  $0 = -\hat{\pi} \log \hat{\pi} + \hat{\pi} \log \hat{\pi} - (1 - \hat{\pi}) \log(1 - \hat{\pi}) + (1 - \hat{\pi}) \log(1 - \hat{\pi})$ , this can be rewritten as  $\hat{\mathcal{L}}(\mathcal{G}, q) = \hat{\pi} \log \frac{\hat{\pi}}{\pi} + (1 - \hat{\pi}) \log \frac{1-\hat{\pi}}{1-\pi} - \hat{\pi} \log \frac{\hat{\pi}}{q} - (1 - \hat{\pi}) \log \frac{1-\hat{\pi}}{1-q} - n \log 2$ , and by using eq.(5) we prove our claim.

Note that by maximizing with respect to the mixture parameter  $q$  and by properties of the KL divergence, we get  $KL(\hat{\pi} \| \hat{q}) = 0 \Leftrightarrow \hat{q} = \hat{\pi}$ . We define our hypothesis space  $\Upsilon$  given the conditions in Remark 2 and Propositions 4 and 5. For the case  $\hat{\pi} = 1$ , we “shrink” the optimal mixture parameter  $\hat{q}$  to  $1 - \frac{1}{2m}$  in order to avoid generating an invalid PMF as discussed in Remark 2.  $\square$

**Remark 8.** *Recall that a trivial game (e.g.  $\mathcal{G} = (\mathbf{W}, \mathbf{b})$ ,  $\mathbf{W} = \mathbf{0}$ ,  $\mathbf{b} = \mathbf{0}$ ,  $\pi(\mathcal{G}) = 1$ ) induces a uniform PMF by Remark 2, and therefore its log-likelihood is  $-n \log 2$ . Note that the lowest log-likelihood for non-trivial identifiable games in eq.(7) is  $-n \log 2$  by setting the optimal mixture parameter  $\hat{q} = \hat{\pi}(\mathcal{G})$  and given that  $KL(\hat{\pi}(\mathcal{G}) \| \pi(\mathcal{G})) \geq 0$ .*

Furthermore, eq.(7) implies that for non-trivial identifiable games  $\mathcal{G}$ , we expect the *true proportion of equilibria*  $\pi(\mathcal{G})$  to be strictly less than the *empirical proportion of equilibria*  $\hat{\pi}(\mathcal{G})$  in the given data. This is by setting the optimal mixture parameter  $\hat{q} = \hat{\pi}(\mathcal{G})$  and the condition  $q > \pi(\mathcal{G})$  in our hypothesis space.

## 5 Generalization Bound and VC-Dimension

In this section, we show a generalization bound for the maximum likelihood problem as well as an upper bound of the VC-dimension of linear influence games. Our objective is to establish that with probability at least  $1 - \delta$ , for some confidence parameter  $0 < \delta < 1$ , the maximum likelihood estimate is within  $\epsilon > 0$  of the optimal parameters, in terms of achievable expected log-likelihood.

Given the ground truth distribution  $\mathcal{Q}$  of the data, let  $\bar{\pi}(\mathcal{G})$  be the *expected proportion of equilibria*, i.e.:

$$\bar{\pi}(\mathcal{G}) = \mathbb{P}_{\mathcal{Q}}[\mathbf{x} \in \mathcal{NE}(\mathcal{G})] \quad (8)$$

and let  $\bar{\mathcal{L}}(\mathcal{G}, q)$  be the *expected log-likelihood* of a generative model from game  $\mathcal{G}$  and mixture parameter  $q$ , i.e.:

$$\bar{\mathcal{L}}(\mathcal{G}, q) = \mathbb{E}_{\mathcal{Q}}[\log p_{(\mathcal{G}, q)}(\mathbf{x})] \quad (9)$$

Note that our hypothesis space  $\Upsilon$  in eq.(7) includes a continuous parameter  $q$  that could potentially have infinite VC-dimension. The following lemma will allow us later to prove that uniform convergence for the extreme values of  $q$  implies uniform convergence for all  $q$  in the domain.

**Lemma 9.** *Consider any game  $\mathcal{G}$  and, for  $0 < q'' < q' < q < 1$ , let  $\theta = (\mathcal{G}, q)$ ,  $\theta' = (\mathcal{G}, q')$  and  $\theta'' = (\mathcal{G}, q'')$ . If, for any  $\epsilon > 0$  we have  $|\hat{\mathcal{L}}(\theta) - \bar{\mathcal{L}}(\theta)| \leq \epsilon/2$  and  $|\hat{\mathcal{L}}(\theta'') - \bar{\mathcal{L}}(\theta'')| \leq \epsilon/2$ , then  $|\hat{\mathcal{L}}(\theta') - \bar{\mathcal{L}}(\theta')| \leq \epsilon/2$ .*

*Proof.* Let  $\mathcal{NE} \equiv \mathcal{NE}(\mathcal{G})$ ,  $\pi \equiv \pi(\mathcal{G})$ ,  $\hat{\pi} \equiv \hat{\pi}(\mathcal{G})$ ,  $\bar{\pi} \equiv \bar{\pi}(\mathcal{G})$ , and  $\mathbb{E}[\cdot]$  and  $\mathbb{P}[\cdot]$  be the expectation and probability with respect to the ground truth distribution  $\mathcal{Q}$  of the data.

First note that for any  $\theta = (\mathcal{G}, q)$ , we have  $\bar{\mathcal{L}}(\theta) = \mathbb{E}[\log p_{(\mathcal{G}, q)}(\mathbf{x})] = \mathbb{E}[1[\mathbf{x} \in \mathcal{NE}] \log \frac{q}{|\mathcal{NE}|} + 1[\mathbf{x} \notin \mathcal{NE}] \log \frac{1-q}{2^n - |\mathcal{NE}|}] = \mathbb{P}[\mathbf{x} \in \mathcal{NE}] \log \frac{q}{|\mathcal{NE}|} + \mathbb{P}[\mathbf{x} \notin \mathcal{NE}] \log \frac{1-q}{2^n - |\mathcal{NE}|} = \bar{\pi} \log \frac{q}{|\mathcal{NE}|} + (1 - \bar{\pi}) \log \frac{1-q}{2^n - |\mathcal{NE}|} = \bar{\pi} \log \left( \frac{q}{1-q} \cdot \frac{2^n - |\mathcal{NE}|}{|\mathcal{NE}|} \right) + \log \frac{1-q}{2^n - |\mathcal{NE}|} = \bar{\pi} \log \left( \frac{q}{1-q} \cdot \frac{1-\pi}{\pi} \right) + \log \frac{1-q}{1-\pi} - n \log 2$ .

Similarly, for any  $\theta = (\mathcal{G}, q)$ , we have  $\hat{\mathcal{L}}(\theta) = \hat{\pi} \log \left( \frac{q}{1-q} \cdot \frac{1-\pi}{\pi} \right) + \log \frac{1-q}{1-\pi} - n \log 2$ . So that  $\hat{\mathcal{L}}(\theta) - \bar{\mathcal{L}}(\theta) = (\hat{\pi} - \bar{\pi}) \log \left( \frac{q}{1-q} \cdot \frac{1-\pi}{\pi} \right)$ .

Furthermore, the function  $\frac{q}{1-q}$  is strictly monotonically increasing for  $0 \leq q < 1$ . If  $\hat{\pi} > \bar{\pi}$  then  $-\epsilon/2 \leq \hat{\mathcal{L}}(\theta'') - \bar{\mathcal{L}}(\theta'') < \hat{\mathcal{L}}(\theta') - \bar{\mathcal{L}}(\theta') < \hat{\mathcal{L}}(\theta) - \bar{\mathcal{L}}(\theta) \leq \epsilon/2$ . Else, if  $\hat{\pi} < \bar{\pi}$ , we have  $\epsilon/2 \geq \hat{\mathcal{L}}(\theta'') - \bar{\mathcal{L}}(\theta'') > \hat{\mathcal{L}}(\theta') - \bar{\mathcal{L}}(\theta') > \hat{\mathcal{L}}(\theta) - \bar{\mathcal{L}}(\theta) \geq -\epsilon/2$ . Finally, if  $\hat{\pi} = \bar{\pi}$  then  $\hat{\mathcal{L}}(\theta'') - \bar{\mathcal{L}}(\theta'') = \hat{\mathcal{L}}(\theta') - \bar{\mathcal{L}}(\theta') = \hat{\mathcal{L}}(\theta) - \bar{\mathcal{L}}(\theta) = 0$ .  $\square$

The following theorem shows that the expected log-likelihood of the maximum likelihood estimate converges in probability to that of the optimal, as the data size  $m$  increases.



**Theorem 10.** Let  $\hat{\theta} = (\hat{\mathcal{G}}, \hat{q})$  be the maximum likelihood estimate in eq.(7) and  $\bar{\theta} = (\bar{\mathcal{G}}, \bar{q})$  be the maximum expected likelihood estimate, i.e.  $\hat{\theta} = \arg \max_{\theta \in \Upsilon} \hat{\mathcal{L}}(\theta)$  and  $\bar{\theta} = \arg \max_{\theta \in \Upsilon} \bar{\mathcal{L}}(\theta)$ , then with probability at least  $1 - \delta$ :

$$\bar{\mathcal{L}}(\hat{\theta}) \geq \bar{\mathcal{L}}(\bar{\theta}) - \left( \log \max(2m, \frac{1}{1-\bar{q}}) + n \log 2 \right) \sqrt{\frac{2}{m} (\log d(\mathcal{H}) + \log \frac{4}{\delta})} \quad (10)$$

where  $\mathcal{H}$  is the class of games of interest,  $\Upsilon = \{(\mathcal{G}, q) \mid \mathcal{G} \in \mathcal{H} \wedge 0 < \pi(\mathcal{G}) < q < 1\}$  is the hypothesis space of non-trivial identifiable games and  $d(\mathcal{H}) \equiv |\cup_{\mathcal{G} \in \mathcal{H}} \{\mathcal{NE}(\mathcal{G})\}|$  is the number of all possible games in  $\mathcal{H}$  (identified by their Nash equilibria sets).

*Proof.* First our objective is to find a lower bound for  $\mathbb{P}[\bar{\mathcal{L}}(\hat{\theta}) - \bar{\mathcal{L}}(\bar{\theta}) \geq -\epsilon] \geq \mathbb{P}[\bar{\mathcal{L}}(\hat{\theta}) - \bar{\mathcal{L}}(\bar{\theta}) \geq -\epsilon + (\hat{\mathcal{L}}(\hat{\theta}) - \hat{\mathcal{L}}(\bar{\theta}))] \geq \mathbb{P}[-\hat{\mathcal{L}}(\hat{\theta}) + \bar{\mathcal{L}}(\hat{\theta}) \geq -\frac{\epsilon}{2}, \hat{\mathcal{L}}(\bar{\theta}) - \bar{\mathcal{L}}(\bar{\theta}) \geq -\frac{\epsilon}{2}] = \mathbb{P}[\hat{\mathcal{L}}(\hat{\theta}) - \bar{\mathcal{L}}(\hat{\theta}) \leq \frac{\epsilon}{2}, \hat{\mathcal{L}}(\bar{\theta}) - \bar{\mathcal{L}}(\bar{\theta}) \geq -\frac{\epsilon}{2}] = 1 - \mathbb{P}[\hat{\mathcal{L}}(\hat{\theta}) - \bar{\mathcal{L}}(\hat{\theta}) > \frac{\epsilon}{2} \vee \hat{\mathcal{L}}(\bar{\theta}) - \bar{\mathcal{L}}(\bar{\theta}) < -\frac{\epsilon}{2}].$

Let  $\tilde{q} \equiv \max(1 - \frac{1}{2m}, \bar{q})$ . Now, we have  $\mathbb{P}[\hat{\mathcal{L}}(\hat{\theta}) - \bar{\mathcal{L}}(\hat{\theta}) > \frac{\epsilon}{2} \vee \hat{\mathcal{L}}(\bar{\theta}) - \bar{\mathcal{L}}(\bar{\theta}) < -\frac{\epsilon}{2}] \leq \mathbb{P}[(\exists \theta \in \Upsilon, q \leq \tilde{q}) \mid \hat{\mathcal{L}}(\theta) - \bar{\mathcal{L}}(\theta) \mid > \frac{\epsilon}{2}] = \mathbb{P}[(\exists \theta, \mathcal{G} \in \mathcal{H}, q \in \{\pi(\mathcal{G}), \tilde{q}\}) \mid \hat{\mathcal{L}}(\theta) - \bar{\mathcal{L}}(\theta) \mid > \frac{\epsilon}{2}]$ . The last equality follows from invoking Lemma 9.

Note that  $\mathbb{E}[\hat{\mathcal{L}}(\theta)] = \bar{\mathcal{L}}(\theta)$  and that since  $\pi(\mathcal{G}) \leq q \leq \tilde{q}$ , the log-likelihood is bounded as  $(\forall \mathbf{x}) -B \leq \log p_{(\mathcal{G}, q)}(\mathbf{x}) \leq 0$ , where  $B = \log \frac{1}{1-\tilde{q}} + n \log 2 = \log \max(2m, \frac{1}{1-\bar{q}}) + n \log 2$ .

Therefore, by Hoeffding's inequality, we have  $\mathbb{P}[\mid \hat{\mathcal{L}}(\theta) - \bar{\mathcal{L}}(\theta) \mid > \frac{\epsilon}{2}] \leq 2e^{-\frac{m\epsilon^2}{2B^2}}$ .

Furthermore, note that there are  $2d(\mathcal{H})$  possible parameters  $\theta$ , since we need to consider only two values of  $q \in \{\pi(\mathcal{G}), \tilde{q}\}$  and because the number of all possible games in  $\mathcal{H}$  (identified by their Nash equilibria sets) is  $d(\mathcal{H}) \equiv |\cup_{\mathcal{G} \in \mathcal{H}} \{\mathcal{NE}(\mathcal{G})\}|$ . Therefore, by the union bound we get the following uniform convergence  $\mathbb{P}[(\exists \theta, \mathcal{G} \in \mathcal{H}, q \in \{\pi(\mathcal{G}), \tilde{q}\}) \mid \hat{\mathcal{L}}(\theta) - \bar{\mathcal{L}}(\theta) \mid > \frac{\epsilon}{2}] \leq 4d(\mathcal{H})\mathbb{P}[\mid \hat{\mathcal{L}}(\theta) - \bar{\mathcal{L}}(\theta) \mid > \frac{\epsilon}{2}] \leq 4d(\mathcal{H})e^{-\frac{m\epsilon^2}{2B^2}} = \delta$ . Finally, by solving for  $\delta$  we prove our claim.  $\square$

The following theorem establishes the complexity of the class of linear influence games, which implies that the term  $\log d(\mathcal{H})$  of the generalization bound in Theorem 10 is only polynomial in the number of players  $n$ .

**Theorem 11.** Let  $\mathcal{H}$  be the class of linear influence games. Then  $d(\mathcal{H}) \equiv |\cup_{\mathcal{G} \in \mathcal{H}} \{\mathcal{NE}(\mathcal{G})\}| \leq 2^{n \frac{n(n+1)}{2} + 1} \leq 2^{n^3}$ .

*Proof.* The logarithm of the number of possible pure-strategy Nash equilibria sets supported by  $\mathcal{H}$  (i.e., that can be produced by some game in  $\mathcal{H}$ ) is upper bounded by the VC-dimension of the class of neural networks with a single hidden layer of  $n$  units and  $n + \binom{n}{2}$  input units, linear threshold activation functions, and constant output weights.

For every linear influence game  $\mathcal{G} = (\mathbf{W}, \mathbf{b})$  in  $\mathcal{H}$ , define the neural network with a single layer of  $n$  hidden units,  $n$  of the inputs corresponds to the linear terms  $x_1, \dots, x_n$  and  $\binom{n}{2}$  corresponds to the quadratic polynomial terms  $x_i x_j$  for all pairs of players  $(i, j)$ ,  $1 \leq i < j \leq n$ . For every hidden unit  $i$ , the weights corresponding to the linear terms  $x_1, \dots, x_n$  are  $-b_1, \dots, -b_n$ , respectively, while the weights corresponding to the quadratic terms  $x_i x_j$  are  $-w_{ij}$ , for all pairs of players  $(i, j)$ ,  $1 \leq i < j \leq n$ , respectively. The weights of the bias term of all the hidden units are set to 0. All  $n$  output weights are set to 1 while the weight of the output bias term is set to 0. The output of the neural network

is  $1[\mathbf{x} \notin \mathcal{NE}(\mathcal{G})]$ . Note that we define the neural network to classify non-equilibrium as opposed to equilibrium to keep the convention in the neural network literature to define the threshold function to output 0 for input 0. The alternative is to redefine the threshold function to output 1 instead for input 0.

Finally, we use the VC-dimension of neural networks [Sontag, 1998].  $\square$

From Theorems 10 and 11, we state the generalization bounds for linear influence games.

**Corollary 12.** *Let  $\hat{\theta} = (\hat{\mathcal{G}}, \hat{q})$  be the maximum likelihood estimate in eq.(7) and  $\bar{\theta} = (\bar{\mathcal{G}}, \bar{q})$  be the maximum expected likelihood estimate, i.e.  $\hat{\theta} = \arg \max_{\theta \in \Upsilon} \hat{\mathcal{L}}(\theta)$  and  $\bar{\theta} = \arg \max_{\theta \in \Upsilon} \bar{\mathcal{L}}(\theta)$ , then with probability at least  $1 - \delta$ :*

$$\bar{\mathcal{L}}(\hat{\theta}) \geq \bar{\mathcal{L}}(\bar{\theta}) - \left( \log \max(2m, \frac{1}{1-\bar{q}}) + n \log 2 \right) \sqrt{\frac{2}{m} (n^3 \log 2 + \log \frac{4}{\delta})} \quad (11)$$

where  $\mathcal{H}$  is the class of linear influence games and  $\Upsilon = \{(\mathcal{G}, q) \mid \mathcal{G} \in \mathcal{H} \wedge 0 < \pi(\mathcal{G}) < q < 1\}$  is the hypothesis space of non-trivial identifiable linear influence games.

## 6 Algorithms

In this section, we approximate the maximum likelihood problem by maximizing the number of observed equilibria in the data, suitable for a hypothesis space of games with small true proportion of equilibria. We then present our convex loss minimization approach. We also discuss baseline methods such as sigmoidal approximation and exhaustive search.

First, we discuss some negative results that justifies the use of simple approaches. Counting the number of Nash equilibria is NP-hard for influence games, and so is computing the log-likelihood function and therefore maximum likelihood estimation. This is not a disadvantage relative to probabilistic graphical models, since computing the log-likelihood function is also NP-hard for Ising models and Markov random fields in general, while learning is also NP-hard for Bayesian networks. General approximation techniques such as pseudo-likelihood estimation do not lead to tractable methods for learning linear influence games. From an optimization perspective, the log-likelihood function is not continuous because of the number of equilibria. Therefore, we cannot rely on concepts such as Lipschitz continuity. Furthermore, bounding the number of equilibria by known bounds for Ising models leads to trivial bounds. (Formal proofs and discussion are included in Appendix A.)

### 6.1 An Exact Quasi-Linear Method for General Games: Sample-Picking

As a first approach, consider solving the maximum likelihood estimation problem in eq.(7) by an exact exhaustive search algorithm. This algorithm iterates through all possible Nash equilibria sets, i.e. for  $s = 0, \dots, 2^n$ , we generate all possible sets of size  $s$  with elements from the joint-action space  $\{-1, +1\}^n$ . Recall that there exists  $\binom{2^n}{s}$  of such sets of size  $s$  and since  $\sum_{s=0}^{2^n} \binom{2^n}{s} = 2^{2^n}$  the search space is super-exponential in the number of players  $n$ .

Based on few observations, we can obtain an  $\mathcal{O}(m \log m)$  algorithm for  $m$  samples. First, note that the above method does not constrain the set of Nash equilibria in any fashion. Therefore, only joint-actions that are observed in the data are candidates of being Nash

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**Algorithm 1** Sample-Picking for General Games

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**Input:** Dataset  $\mathcal{D} = \mathbf{x}^{(1)}, \dots, \mathbf{x}^{(m)}$   
 Compute the unique samples  $\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(U)}$  and their frequency  $\hat{p}^{(1)}, \dots, \hat{p}^{(U)}$  in the dataset  $\mathcal{D}$   
 Sort joint-actions by their frequency such that  $\hat{p}^{(1)} \geq \hat{p}^{(2)} \geq \dots \geq \hat{p}^{(U)}$   
**for** each unique sample  $k = 1, \dots, U$  **do**  
   Define  $\mathcal{G}_k$  by the Nash equilibria set  $\mathcal{NE}(\mathcal{G}_k) = \{\mathbf{y}^{(1)}, \dots, \mathbf{y}^{(k)}\}$   
   Compute the log-likelihood  $\hat{\mathcal{L}}(\mathcal{G}_k, \hat{q}_k)$  in eq.(7) (note that  $\hat{q}_k = \hat{\pi}(\mathcal{G}) = \frac{1}{m}(\hat{p}^{(1)} + \dots + \hat{p}^{(k)})$ ,  
    $\pi(\mathcal{G}) = \frac{k}{2^n}$ )  
**end for**  
**Output:** The game  $\mathcal{G}_{\hat{k}}$  such that  $\hat{k} = \arg \max_k \hat{\mathcal{L}}(\mathcal{G}_k, \hat{q}_k)$

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equilibria in order to maximize the log-likelihood. This is because the introduction of an unobserved joint-action will increase the true proportion of equilibria without increasing the empirical proportion of equilibria and thus leading to a lower log-likelihood in eq.(7). Second, given a fixed number of Nash equilibria  $k$ , the best strategy would be to pick the  $k$  joint-actions that appear more frequently in the observed data. This will maximize the empirical proportion of equilibria, which will maximize the log-likelihood. Based on these observations, we propose Algorithm 1.

As an aside note, the fact that general games do not constrain the set of Nash equilibria, makes the method more likely to over-fit. On the other hand, influence games will potentially include unobserved equilibria given the linearity constraints in the search space, and thus they would be less likely to over-fit.

## 6.2 An Exact Super-Exponential Method for Influence Games: Exhaustive Search

Note that in the previous subsection, we search in the space of all possible games, not only the linear influence games. First note that *sample-picking* for linear games is NP-hard, i.e. at any iteration of *sample-picking*, checking whether the set of Nash equilibria  $\mathcal{NE}$  corresponds to an influence game or not is equivalent to the following constraint satisfaction problem with linear constraints:

$$\begin{aligned} & \min_{\mathbf{W}, \mathbf{b}} 1 \\ \text{s.t. } & (\forall \mathbf{x} \in \mathcal{NE}) \ x_1(\mathbf{w}_{1,-1}^T \mathbf{x}_{-1} - b_1) \geq 0 \wedge \dots \wedge x_n(\mathbf{w}_{n,-n}^T \mathbf{x}_{-n} - b_n) \geq 0 \\ & (\forall \mathbf{x} \notin \mathcal{NE}) \ x_1(\mathbf{w}_{1,-1}^T \mathbf{x}_{-1} - b_1) < 0 \vee \dots \vee x_n(\mathbf{w}_{n,-n}^T \mathbf{x}_{-n} - b_n) < 0 \end{aligned} \quad (12)$$

Note that eq.(12) contains “or” operators in order to account for the non-equilibria. This makes the problem of finding the  $(\mathbf{W}, \mathbf{b})$  that satisfies such conditions NP-hard for a non-empty complement set  $\{-1, +1\}^n - \mathcal{NE}$ . Furthermore, since *sample-picking* only consider observed equilibria, the search is not optimal with respect to the space of influence games.

Regarding a more refined approach for enumerating influence games only, note that in an influence game each player separates hypercube vertices with a linear function, i.e. for  $\mathbf{v} \equiv (\mathbf{w}_{i,-i}, b_i)$  and  $\mathbf{y} \equiv (x_i \mathbf{x}_{-i}, -x_i) \in \{-1, +1\}^n$  we have  $x_i(\mathbf{w}_{i,-i}^T \mathbf{x}_{-i} - b_i) = \mathbf{v}^T \mathbf{y}$ . Assume we assign a binary label to each vertex  $\mathbf{y}$ , then note that not all possible labelings are linearly separable. Labelings which are linearly separable are called *linear threshold*

functions (LTFs). A lower bound of the number of LTFs was first provided in Muroga [1965], which showed that the number of LTFs is at least  $\alpha(n) \equiv 2^{0.33048n^2}$ . Tighter lower bounds were shown later in Yamija and Ibaraki [1965] for  $n \geq 6$  and in Muroga and Toda [1966] for  $n \geq 8$ . Regarding an upper bound, Winder [1960] showed that the number of LTFs is at most  $\beta(n) \equiv 2^{n^2}$ . By using such bounds for all players, we can conclude that there is at least  $\alpha(n)^n = 2^{0.33048n^3}$  and at most  $\beta(n)^n = 2^{n^3}$  influence games (which is indeed another upper bound of the VC-dimension of the class of influence games; the bound in Theorem 11 is tighter and uses bounds of the VC-dimension of neural networks). The bounds discussed above would bound the time-complexity of a search algorithm if we could easily enumerate all LTFs for a single player. Unfortunately, this seems to be far from a trivial problem. By using results in Muroga [1971], a weight vector  $\mathbf{v}$  with integer entries such that  $(\forall i) |v_i| \leq \beta(n) \equiv (n+1)^{(n+1)/2}/2^n$  is sufficient to realize all possible LTFs. Therefore we can conclude that enumerating influence games takes at most  $(2\beta(n) + 1)^{n^2} \approx (\frac{\sqrt{n+1}}{2})^{n^3}$  steps, and we propose the use of this method only for  $n \leq 4$ .

For  $n = 4$  we found that the number of influence games is 23,706. Experimentally, we did not find differences between this method and *sample-picking* since most of the time, the model with maximum likelihood was an influence game.

### 6.3 From Maximum Likelihood to Maximum Empirical Proportion of Equilibria

We approximately perform maximum likelihood estimation for influence games, by maximizing the *empirical proportion of equilibria*, i.e. the equilibria in the observed data. This strategy allows us to avoid computing  $\pi(\mathcal{G})$  as in eq.(4) for maximum likelihood estimation (given its dependence on  $|\mathcal{NE}(\mathcal{G})|$ ). We propose this approach for games with small true proportion of equilibria with high probability, i.e. with probability at least  $1 - \delta$ , we have  $\pi(\mathcal{G}) \leq \frac{\kappa^n}{\delta}$  for  $0 < \kappa < 1$ . Particularly, we will show in Section 7 that for influence games we have  $\kappa = 3/4$ . Given this, our approximate problem relies on a bound of the log-likelihood that holds with high probability. We also show that under very mild conditions, the parameters  $(\mathcal{G}, q)$  belong to the hypothesis space of the original problem with high probability.

First, we derive bounds on the log-likelihood function.

**Lemma 13.** *Given a non-trivial game  $\mathcal{G}$  with  $0 < \pi(\mathcal{G}) < \hat{\pi}(\mathcal{G})$ , the KL divergence in the log-likelihood function in eq.(7) is bounded as follows:*

$$-\hat{\pi}(\mathcal{G}) \log \pi(\mathcal{G}) - \log 2 < KL(\hat{\pi}(\mathcal{G}) \parallel \pi(\mathcal{G})) < -\hat{\pi}(\mathcal{G}) \log \pi(\mathcal{G}) \quad (13)$$

*Proof.* Let  $\pi \equiv \pi(\mathcal{G})$  and  $\hat{\pi} \equiv \hat{\pi}(\mathcal{G})$ . Note that  $\alpha(\pi) \equiv \lim_{\hat{\pi} \rightarrow 0} KL(\hat{\pi} \parallel \pi) = 0$  and  $\beta(\pi) \equiv \lim_{\hat{\pi} \rightarrow 1} KL(\hat{\pi} \parallel \pi) = -\log \pi \leq n \log 2$ . Since the function is convex we can upper-bound it by  $\alpha(\pi) + (\beta(\pi) - \alpha(\pi))\hat{\pi} = -\hat{\pi} \log \pi$ .

To find a lower bound, we find the point in which the derivative of the original function is equal to the slope of the upper bound, i.e.  $\frac{\partial KL(\hat{\pi} \parallel \pi)}{\partial \hat{\pi}} = \beta(\pi) - \alpha(\pi) = -\log \pi$ , which gives  $\hat{\pi}^* = \frac{1}{2-\pi}$ . Then, the maximum difference between the upper bound and the original function is given by  $\lim_{\pi \rightarrow 0} -\hat{\pi}^* \log \pi - KL(\hat{\pi}^* \parallel \pi) = \log 2$ .  $\square$

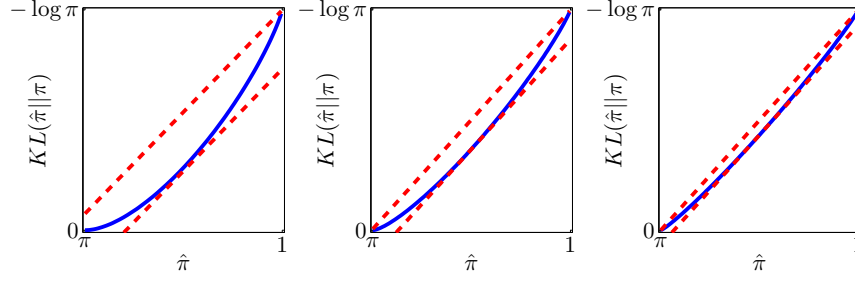


Figure 2: KL divergence (blue) and bounds derived in Lemma 13 (red) for  $\pi = (3/4)^n$  where  $n = 9$  (left),  $n = 18$  (center) and  $n = 36$  (right). Note that the bounds are very informative when  $n \rightarrow +\infty$  (or equivalently when  $\pi \rightarrow 0$ ).

Note that the lower and upper bounds are very informative when  $\pi(\mathcal{G}) \rightarrow 0$  (or in our setting when  $n \rightarrow +\infty$ ), since  $\log 2$  becomes small when compared to  $-\log \pi(\mathcal{G})$ , as shown in Figure 2.

Next, we derive the problem of maximizing the empirical proportion of equilibria from the maximum likelihood estimation problem.

**Theorem 14.** *Assume that with probability at least  $1 - \delta$  we have  $\pi(\mathcal{G}) \leq \frac{\kappa^n}{\delta}$  for  $0 < \kappa < 1$ . Maximizing a lower bound (with high probability) of the log-likelihood in eq.(7) is equivalent to maximizing the empirical proportion of equilibria:*

$$\max_{\mathcal{G} \in \mathcal{H}} \hat{\pi}(\mathcal{G}) \quad (14)$$

furthermore, for all games  $\mathcal{G}$  such that  $\hat{\pi}(\mathcal{G}) \geq \gamma$  for some  $0 < \gamma < 1/2$ , for sufficiently large  $n > \log_{\kappa}(\delta\gamma)$  and optimal mixture parameter  $\hat{q} = \min(\hat{\pi}(\mathcal{G}), 1 - \frac{1}{2m})$ , we have  $(\mathcal{G}, \hat{q}) \in \Upsilon$ , where  $\Upsilon = \{(\mathcal{G}, q) \mid \mathcal{G} \in \mathcal{H} \wedge 0 < \pi(\mathcal{G}) < q < 1\}$  is the hypothesis space of non-trivial identifiable games.

*Proof.* By applying the lower bound in Lemma 13 in eq.(7) to non-trivial games, we have  $\hat{\mathcal{L}}(\mathcal{G}, \hat{q}) = KL(\hat{\pi}(\mathcal{G}) \parallel \pi(\mathcal{G})) - KL(\hat{\pi}(\mathcal{G}) \parallel \hat{q}) - n \log 2 > -\hat{\pi}(\mathcal{G}) \log \pi(\mathcal{G}) - KL(\hat{\pi}(\mathcal{G}) \parallel \hat{q}) - (n+1) \log 2$ . Since  $\pi(\mathcal{G}) \leq \frac{\kappa^n}{\delta}$ , we have  $-\log \pi(\mathcal{G}) \geq -\log \frac{\kappa^n}{\delta}$ . Therefore  $\hat{\mathcal{L}}(\mathcal{G}, \hat{q}) > -\hat{\pi}(\mathcal{G}) \log \frac{\kappa^n}{\delta} - KL(\hat{\pi}(\mathcal{G}) \parallel \hat{q}) - (n+1) \log 2$ . Regarding the term  $KL(\hat{\pi}(\mathcal{G}) \parallel \hat{q})$ , if  $\hat{\pi}(\mathcal{G}) < 1 \Rightarrow KL(\hat{\pi}(\mathcal{G}) \parallel \hat{q}) = KL(\hat{\pi}(\mathcal{G}) \parallel \hat{\pi}(\mathcal{G})) = 0$ , and if  $\hat{\pi}(\mathcal{G}) = 1 \Rightarrow KL(\hat{\pi}(\mathcal{G}) \parallel \hat{q}) = KL(1 \parallel 1 - \frac{1}{2m}) = -\log(1 - \frac{1}{2m}) \leq \log 2$  and approaches 0 when  $m \rightarrow +\infty$ . Maximizing the lower bound of the log-likelihood becomes  $\max_{\mathcal{G} \in \mathcal{H}} \hat{\pi}(\mathcal{G})$  by removing the constant terms that do not depend on  $\mathcal{G}$ .

In order to prove  $(\mathcal{G}, \hat{q}) \in \Upsilon$  we need to prove  $0 < \pi(\mathcal{G}) < \hat{q} < 1$ . For proving the first inequality  $0 < \pi(\mathcal{G})$ , note that  $\hat{\pi}(\mathcal{G}) \geq \gamma > 0$ , and therefore  $\mathcal{G}$  has at least one equilibria. For proving the third inequality  $\hat{q} < 1$ , note that  $\hat{q} = \min(\hat{\pi}(\mathcal{G}), 1 - \frac{1}{2m}) < 1$ . For proving the second inequality  $\pi(\mathcal{G}) < \hat{q}$ , we need to prove  $\pi(\mathcal{G}) < \hat{\pi}(\mathcal{G})$  and  $\pi(\mathcal{G}) < 1 - \frac{1}{2m}$ . Since  $\pi(\mathcal{G}) \leq \frac{\kappa^n}{\delta}$  and  $\gamma \leq \hat{\pi}(\mathcal{G})$ , it suffices to prove  $\frac{(3/4)^n}{\delta} < \gamma \Rightarrow \pi(\mathcal{G}) < \hat{\pi}(\mathcal{G})$ . Similarly we need to prove  $\frac{(3/4)^n}{\delta} < 1 - \frac{1}{2m} \Rightarrow \pi(\mathcal{G}) < 1 - \frac{1}{2m}$ . Putting both together, we have  $\frac{(3/4)^n}{\delta} < \min(\gamma, 1 - \frac{1}{2m}) = \gamma$  since  $\gamma < 1/2$  and  $1 - \frac{1}{2m} \geq 1/2$ . Finally,  $\frac{(3/4)^n}{\delta} < \gamma \Leftrightarrow n > \log_{\kappa}(\delta\gamma)$ .  $\square$

## 6.4 A Non-Concave Maximization Method: Sigmoidal Approximation

A very simple optimization approach can be devised by using a sigmoid in order to approximate the 0/1 function  $1[z \geq 0]$  in the maximum likelihood problem of eq.(7) as well as when maximizing the empirical proportion of equilibria as in eq.(14). We use the following sigmoidal approximation:

$$1[z \geq 0] \approx H_{\alpha,\beta}(z) \equiv \frac{1}{2}(1 + \tanh(\frac{z}{\beta} - \operatorname{arctanh}(1 - 2\alpha^{1/n}))) \quad (15)$$

The additional term  $\alpha$  ensures that for  $\mathcal{G} = (\mathbf{W}, \mathbf{b})$ ,  $\mathbf{W} = \mathbf{0}$ ,  $\mathbf{b} = \mathbf{0}$  we get  $1[\mathbf{x} \in \mathcal{NE}(\mathcal{G})] \approx H_{\alpha,\beta}(0)^n = \alpha$ . We perform gradient ascent on these objective functions that have many local maxima. Note that when maximizing the “sigmoidal” likelihood, each step of the gradient ascent is NP-hard due to the “sigmoidal” true proportion of equilibria. Therefore, we propose the use of the sigmoidal maximum likelihood only for  $n \leq 15$ .

In our implementation, we add an  $\ell_1$ -norm regularizer  $-\rho\|\mathbf{W}\|_1$  where  $\rho > 0$  to both maximization problems. The  $\ell_1$ -norm regularizer encourages sparseness and attempts to lower the generalization error by controlling over-fitting.

## 6.5 Our Proposed Approach: Convex Loss Minimization

From an optimization perspective, it is more convenient to minimize a convex objective instead of a sigmoidal approximation in order to avoid the many local minima.

Note that maximizing the empirical proportion of equilibria in eq.(14) is equivalent to minimizing the empirical proportion of non-equilibria, i.e.  $\min_{\mathcal{G} \in \mathcal{H}} (1 - \hat{\pi}(\mathcal{G}))$ . Furthermore,  $1 - \hat{\pi}(\mathcal{G}) = \frac{1}{m} \sum_l 1[\mathbf{x}^{(l)} \notin \mathcal{NE}(\mathcal{G})]$ . Denote by  $\ell$  the 0/1 loss, i.e.  $\ell(z) = 1[z < 0]$ . For influence games, maximizing the empirical proportion of equilibria in eq.(14) is equivalent to solving the loss minimization problem:

$$\min_{\mathbf{W}, \mathbf{b}} \frac{1}{m} \sum_l \max_i \ell(x_i^{(l)}(\mathbf{w}_{i,-i}^T \mathbf{x}_{-i}^{(l)} - b_i)) \quad (16)$$

We can further relax this problem by introducing convex upper bounds of the 0/1 loss. Note that the use of convex losses also avoids the trivial solution of eq.(16), i.e.  $\mathbf{W} = \mathbf{0}$ ,  $\mathbf{b} = \mathbf{0}$  (which obtains the lowest log-likelihood as discussed in Remark 8). Intuitively speaking, note that minimizing the logistic loss  $\ell(z) = \log(1 + e^{-z})$  will make  $z \rightarrow +\infty$ , while minimizing the hinge loss  $\ell(z) = \max(0, 1 - z)$  will make  $z \rightarrow 1$  unlike the 0/1 loss  $\ell(z) = 1[z < 0]$  that only requires  $z = 0$  in order to be minimized. In what follows, we develop four efficient methods for solving eq.(16) under specific choices of loss functions, i.e. hinge and logistic.

In our implementation, we add an  $\ell_1$ -norm regularizer  $\rho\|\mathbf{W}\|_1$  where  $\rho > 0$  to all the minimization problems. The  $\ell_1$ -norm regularizer encourages sparseness and attempts to lower the generalization error by controlling over-fitting.

**Independent Support Vector Machines and Logistic Regression.** We can relax the loss minimization problem in eq.(16) by using the loose bound  $\max_i \ell(z_i) \leq \sum_i \ell(z_i)$ . This relaxation simplifies the original problem into several independent problems. For each player  $i$ , we train the weights  $(\mathbf{w}_{i,-i}, b_i)$  in order to predict independent (disjoint)

actions. This leads to *1-norm SVMs* of Bradley and Mangasarian [1998], Zhu et al. [2003] and  $\ell_1$ -regularized logistic regression. We solve the latter with the  *$\ell_1$ -projection method* of Schmidt et al. [2007a]. While the training is independent, our goal is not the prediction for independent players but the characterization of joint-actions. The use of these well known techniques in our context is novel, since we interpret the output of SVMs and logistic regression as the parameters of an influence game. Therefore, we use the parameters to measure empirical and true proportion of equilibria, KL divergence and log-likelihood in our probabilistic model.

**Simultaneous Support Vector Machines.** While converting the loss minimization problem in eq.(16) by using loose bounds allow to obtain several independent problems with small number of variables, a second reasonable strategy would be to use tighter bounds at the expense of obtaining a single optimization problem with a higher number of variables.

For the hinge loss  $\ell(z) = \max(0, 1 - z)$ , we have  $\max_i \ell(z_i) = \max(0, 1 - z_1, \dots, 1 - z_n)$  and the loss minimization problem in eq.(16) becomes the following primal linear program:

$$\begin{aligned} \min_{\mathbf{W}, \mathbf{b}, \boldsymbol{\xi}} \quad & \frac{1}{m} \sum_l \xi_l + \rho \|\mathbf{W}\|_1 \\ \text{s.t.} \quad & (\forall l, i) \ x_i^{(l)} (\mathbf{w}_{i,-i}^T \mathbf{x}_{-i}^{(l)} - b_i) \geq 1 - \xi_l \quad , \quad (\forall l) \ \xi_l \geq 0 \end{aligned} \quad (17)$$

where  $\rho > 0$ .

Note that eq.(17) is equivalent to a linear program since we can set  $\mathbf{W} = \mathbf{W}^+ - \mathbf{W}^-$ ,  $\|\mathbf{W}\|_1 = \sum_{ij} w_{ij}^+ + w_{ij}^-$  and add the constraints  $\mathbf{W}^+ \geq \mathbf{0}$  and  $\mathbf{W}^- \geq \mathbf{0}$ . We follow the regular SVM derivation by adding slack variables  $\xi_l$  for each sample  $l$ . This problem is a generalization of *1-norm SVMs* of Bradley and Mangasarian [1998], Zhu et al. [2003].

By Lagrangian duality, the dual of the problem in eq.(17) is the following linear program:

$$\begin{aligned} \max_{\boldsymbol{\alpha}} \quad & \sum_{li} \alpha_{li} \\ \text{s.t.} \quad & (\forall i) \ \left\| \sum_l \alpha_{li} x_i^{(l)} \mathbf{x}_{-i}^{(l)} \right\|_{\infty} \leq \rho \quad , \quad (\forall l, i) \ \alpha_{li} \geq 0 \\ & (\forall i) \ \sum_l \alpha_{li} x_i^{(l)} = 0 \quad , \quad (\forall l) \ \sum_i \alpha_{li} \leq \frac{1}{m} \end{aligned} \quad (18)$$

Furthermore, strong duality holds in this case. Note that eq.(18) is equivalent to a linear program since we can transform the constraint  $\|\mathbf{c}\|_{\infty} \leq \rho$  into  $-\rho \mathbf{1} \leq \mathbf{c} \leq \rho \mathbf{1}$ .

**Simultaneous Logistic Regression.** For the logistic loss  $\ell(z) = \log(1 + e^{-z})$ , we could use the non-smooth loss  $\max_i \ell(z_i)$  directly. Instead, we chose a smooth upper bound, i.e.  $\log(1 + \sum_i e^{-z_i})$  (Discussion is included in Appendix B.) The loss minimization problem in eq.(16) becomes:

$$\min_{\mathbf{W}, \mathbf{b}} \quad \frac{1}{m} \sum_l \log(1 + \sum_i e^{-x_i^{(l)} (\mathbf{w}_{i,-i}^T \mathbf{x}_{-i}^{(l)} - b_i)}) + \rho \|\mathbf{W}\|_1 \quad (19)$$

where  $\rho > 0$ .

In our implementation, we use the  $\ell_1$ -projection method of Schmidt et al. [2007a] for optimizing eq.(19). This method performs a *limited memory Broyden-Fletcher-Goldfarb-Shanno* (L-BFGS) step in an expanded model (i.e.  $\mathbf{W} = \mathbf{W}^+ - \mathbf{W}^-$ ,  $\|\mathbf{W}\|_1 = \sum_{ij} w_{ij}^+ + w_{ij}^-$ ) followed by a projection onto the non-negative orthant to enforce  $\mathbf{W}^+ \geq \mathbf{0}$  and  $\mathbf{W}^- \geq \mathbf{0}$ .

## 7 True Proportion of Equilibria

In this section, we justify the use of convex loss minimization for learning the structure and parameters of influence games. We define *absolute indifference* of players and show that our convex loss minimization approach produces games in which all players are non-absolutely-indifferent. We then provide a bound of the true proportion of equilibria with high probability. Our bound only assumes independence of weight vectors among players. Our bound is distribution-free, i.e. we do not assume a specific distribution for the weight vector of each player. Furthermore, we do not assume any connectivity properties of the underlying graph.

Parallel to our analysis, Daskalakis et al. [2011] analyzed a different setting: random games which structure is drawn from the Erdős-Rényi model (i.e. each edge is present independently with the same probability  $p$ ) and utility functions which are random tables. The analysis in Daskalakis et al. [2011], while more general than ours (which only focus on influence games), it is at the same time more restricted since it assumes either the Erdős-Rényi model for random structures or connectivity properties for deterministic structures.

### 7.1 Convex Loss Minimization Produces Non-Absolutely-Indifferent Players

First, we define the notion of *absolute indifference* of players. Our goal in this subsection is to show that our proposed convex loss algorithms produce influence games in which all players are non-absolutely-indifferent and therefore every player defines constraints to the true proportion of equilibria.

**Definition 15.** *Given an influence game  $\mathcal{G} = (\mathbf{W}, \mathbf{b})$ , we say a player  $i$  is absolutely indifferent if and only if  $(\mathbf{w}_{i,-i}, b_i) = \mathbf{0}$ , and non-absolutely-indifferent if and only if  $(\mathbf{w}_{i,-i}, b_i) \neq \mathbf{0}$ .*

Next, we concentrate on the first ingredient for our bound of the true proportion of equilibria. We show that independent and simultaneous SVM and logistic regression produce games in which all players are non-absolutely-indifferent except for some “degenerate” cases. The following lemma applies to independent SVMs for  $c^{(l)} = 0$  and simultaneous SVMs for  $c^{(l)} = \max(0, \max_{j \neq i} (1 - x_j^{(l)} (\mathbf{w}_{i,-i}^T \mathbf{x}_{-i}^{(l)} - b_i)))$ .

**Lemma 16.** *Given  $(\forall l) c^{(l)} \geq 0$ , the minimization of the hinge training loss  $\widehat{\ell}(\mathbf{w}_{i,-i}, b_i) = \frac{1}{m} \sum_l \max(c^{(l)}, 1 - x_i^{(l)} (\mathbf{w}_{i,-i}^T \mathbf{x}_{-i}^{(l)} - b_i))$  guarantees non-absolutely-indifference of player  $i$  except for some “degenerate” cases, i.e. the optimal solution  $(\mathbf{w}_{i,-i}^*, b_i^*) = \mathbf{0}$  if and only if  $(\forall j \neq i) \sum_l 1[x_i^{(l)} x_j^{(l)} = 1] u^{(l)} = \sum_l 1[x_i^{(l)} x_j^{(l)} = -1] u^{(l)}$  and  $\sum_l 1[x_i^{(l)} = 1] u^{(l)} = \sum_l 1[x_i^{(l)} = -1] u^{(l)}$  where  $u^{(l)}$  is defined as  $c^{(l)} > 1 \Leftrightarrow u^{(l)} = 0$ ,  $c^{(l)} < 1 \Leftrightarrow u^{(l)} = 1$  and  $c^{(l)} = 1 \Leftrightarrow u^{(l)} \in [0; 1]$ .*



*Proof.* Let  $f_i(\mathbf{x}_{-i}) \equiv \mathbf{w}_{i,-i}^T \mathbf{x}_{-i} - b_i$ . By noting that  $\max(\alpha, \beta) = \max_{0 \leq u \leq 1} (\alpha + u(\beta - \alpha))$ , we can rewrite  $\widehat{\ell}(\mathbf{w}_{i,-i}, b_i) = \frac{1}{m} \sum_l \max_{0 \leq u^{(l)} \leq 1} (c^{(l)} + u^{(l)}(1 - x_i^{(l)} f_i(\mathbf{x}_{-i}^{(l)}) - c^{(l)}))$ .

Note that  $\widehat{\ell}$  has the minimizer  $(\mathbf{w}_{i,-i}^*, b_i^*) = \mathbf{0}$  if and only if  $\mathbf{0}$  belongs to the subdifferential set of the non-smooth function  $\widehat{\ell}$  at  $(\mathbf{w}_{i,-i}, b_i) = \mathbf{0}$ . In order to maximize  $\widehat{\ell}$ , we have  $c^{(l)} > 1 - x_i^{(l)} f_i(\mathbf{x}_{-i}^{(l)}) \Leftrightarrow u^{(l)} = 0$ ,  $c^{(l)} < 1 - x_i^{(l)} f_i(\mathbf{x}_{-i}^{(l)}) \Leftrightarrow u^{(l)} = 1$  and  $c^{(l)} = 1 - x_i^{(l)} f_i(\mathbf{x}_{-i}^{(l)}) \Leftrightarrow u^{(l)} \in [0; 1]$ . The previous rules simplify at the solution under analysis, since  $(\mathbf{w}_{i,-i}, b_i) = \mathbf{0} \Rightarrow f_i(\mathbf{x}_{-i}^{(l)}) = 0$ .

Let  $g_j(\mathbf{w}_{i,-i}, b_i) \equiv \frac{\partial \widehat{\ell}}{\partial w_{ij}}(\mathbf{w}_{i,-i}, b_i)$  and  $h(\mathbf{w}_{i,-i}, b_i) \equiv \frac{\partial \widehat{\ell}}{\partial b_i}(\mathbf{w}_{i,-i}, b_i)$ . By making  $(\forall j \neq i) 0 \in g_j(\mathbf{0}, 0)$  and  $0 \in h(\mathbf{0}, 0)$ , we get  $(\forall j \neq i) \sum_l x_i^{(l)} x_j^{(l)} u^{(l)} = 0$  and  $\sum_l x_i^{(l)} u^{(l)} = 0$ . Finally, by noting that  $x_i^{(l)} \in \{-1, 1\}$ , we prove our claim.  $\square$

**Remark 17.** Note that for independent SVMs, the “degenerate” cases in Lemma 16 simplify to  $(\forall j \neq i) \sum_l 1[x_i^{(l)} x_j^{(l)} = 1] = \frac{m}{2}$  and  $\sum_l 1[x_i^{(l)} = 1] = \frac{m}{2}$ .

The following lemma applies to independent logistic regression for  $c^{(l)} = 0$  and simultaneous logistic regression for  $c^{(l)} = \sum_{j \neq i} e^{-x_j^{(l)}(\mathbf{w}_{i,-i}^T \mathbf{x}_{-i}^{(l)} - b_i)}$ .

**Lemma 18.** Given  $(\forall l) c^{(l)} \geq 0$ , the minimization of the logistic training loss  $\widehat{\ell}(\mathbf{w}_{i,-i}, b_i) = \frac{1}{m} \sum_l \log(c^{(l)} + 1 + e^{-x_i^{(l)}(\mathbf{w}_{i,-i}^T \mathbf{x}_{-i}^{(l)} - b_i)})$  guarantees non-absolutely-indifference of player  $i$  except for some “degenerate” cases, i.e. the optimal solution  $(\mathbf{w}_{i,-i}^*, b_i^*) = \mathbf{0}$  if and only if  $(\forall j \neq i) \sum_l \frac{1[x_i^{(l)} x_j^{(l)} = 1]}{c^{(l)+2}} = \sum_l \frac{1[x_i^{(l)} x_j^{(l)} = -1]}{c^{(l)+2}}$  and  $\sum_l \frac{1[x_i^{(l)} = 1]}{c^{(l)+2}} = \sum_l \frac{1[x_i^{(l)} = -1]}{c^{(l)+2}}$ .

*Proof.* Note that  $\widehat{\ell}$  has the minimizer  $(\mathbf{w}_{i,-i}^*, b_i^*) = \mathbf{0}$  if and only if the gradient of the smooth function  $\widehat{\ell}$  is  $\mathbf{0}$  at  $(\mathbf{w}_{i,-i}, b_i) = \mathbf{0}$ . Let  $g_j(\mathbf{w}_{i,-i}, b_i) \equiv \frac{\partial \widehat{\ell}}{\partial w_{ij}}(\mathbf{w}_{i,-i}, b_i)$  and  $h(\mathbf{w}_{i,-i}, b_i) \equiv \frac{\partial \widehat{\ell}}{\partial b_i}(\mathbf{w}_{i,-i}, b_i)$ . By making  $(\forall j \neq i) g_j(\mathbf{0}, 0) = 0$  and  $h(\mathbf{0}, 0) = 0$ , we get  $(\forall j \neq i) \sum_l \frac{x_i^{(l)} x_j^{(l)}}{c^{(l)+2}} = 0$  and  $\sum_l \frac{x_i^{(l)}}{c^{(l)+2}} = 0$ . Finally, by noting that  $x_i^{(l)} \in \{-1, 1\}$ , we prove our claim.  $\square$

**Remark 19.** Note that for independent logistic regression, the “degenerate” cases in Lemma 18 simplify to  $(\forall j \neq i) \sum_l 1[x_i^{(l)} x_j^{(l)} = 1] = \frac{m}{2}$  and  $\sum_l 1[x_i^{(l)} = 1] = \frac{m}{2}$ .

Based on these results, after termination of our proposed algorithms, we fix cases in which the optimal solution  $(\mathbf{w}_{i,-i}^*, b_i^*) = \mathbf{0}$  by setting  $b_i^* = 1$  if the action of player  $i$  was mostly  $-1$  or  $b_i^* = -1$  otherwise. We point out to the careful reader that we did not include the  $\ell_1$ -regularization term in the above proofs since the subdifferential of  $\rho \|\mathbf{w}_{i,-i}\|_1$  vanishes at  $\mathbf{w}_{i,-i} = \mathbf{0}$ , and therefore our proofs still hold.

## 7.2 Bounding the True Proportion of Equilibria

In what follows, we concentrate on the second ingredient for our bound of the true proportion of equilibria. We show that for a game with a single *non-absolutely-indifferent* player, the true proportion of equilibria is bounded by  $3/4$ .

**Lemma 20.** *Given an influence game  $\mathcal{G} = (\mathbf{W}, \mathbf{b})$  with non-absolutely-indifferent player  $i$  and absolutely-indifferent players  $\forall j \neq i$ , the following statements hold:*

$$\begin{aligned} \text{i. } & \mathbf{x} \in \mathcal{NE}(\mathcal{G}) \Leftrightarrow x_i(\mathbf{w}_{i,-i}^T \mathbf{x}_{-i} - b_i) \geq 0 \\ \text{ii. } & |\mathcal{NE}(\mathcal{G})| = 2^{n-1} + \sum_{\mathbf{x}_{-i}} 1[\mathbf{w}_{i,-i}^T \mathbf{x}_{-i} - b_i = 0] \\ \text{iii. } & \frac{1}{2} \leq \pi(\mathcal{G}) \leq \frac{3}{4} \end{aligned} \quad (20)$$

*Proof.* Let  $f_i(\mathbf{x}_{-i}) \equiv \mathbf{w}_{i,-i}^T \mathbf{x}_{-i} - b_i$ . For proving Claim i, note that  $1[\mathbf{x} \in \mathcal{NE}(\mathcal{G})] = \min_j 1[x_j f_j(\mathbf{x}_{-j}) \geq 0] = 1[x_i f_i(\mathbf{x}_{-i}) \geq 0] \min_{j \neq i} 1[x_j f_j(\mathbf{x}_{-j}) \geq 0]$ . Since all players except  $i$  are absolutely-indifferent, we have  $(\forall j \neq i) (\mathbf{w}_{j,-j}, b_j) = \mathbf{0} \Rightarrow f_j(\mathbf{x}_{-j}) = 0$  which implies that  $\min_{j \neq i} 1[x_j f_j(\mathbf{x}_{-j}) \geq 0] = 1$ . Therefore,  $1[\mathbf{x} \in \mathcal{NE}(\mathcal{G})] = 1[x_i f_i(\mathbf{x}_{-i}) \geq 0]$ .

For proving Claim ii, by Claim i we have  $|\mathcal{NE}(\mathcal{G})| = \sum_{\mathbf{x}} 1[x_i f_i(\mathbf{x}_{-i}) \geq 0]$ . We can rewrite  $|\mathcal{NE}(\mathcal{G})| = \sum_{\mathbf{x}} 1[x_i = +1]1[f_i(\mathbf{x}_{-i}) \geq 0] + \sum_{\mathbf{x}} 1[x_i = -1]1[f_i(\mathbf{x}_{-i}) \leq 0]$  or equivalently  $|\mathcal{NE}(\mathcal{G})| = \sum_{\mathbf{x}_{-i}} 1[f_i(\mathbf{x}_{-i}) \geq 0] + \sum_{\mathbf{x}_{-i}} 1[f_i(\mathbf{x}_{-i}) \leq 0] = 2^{n-1} + \sum_{\mathbf{x}_{-i}} 1[f_i(\mathbf{x}_{-i}) = 0]$ .

For proving Claim iii, by eq.(4) and Claim ii we have  $\pi(\mathcal{G}) = \frac{|\mathcal{NE}(\mathcal{G})|}{2^n} = \frac{1}{2} + \frac{1}{2^n} \alpha(\mathbf{w}_{i,-i}, b_i)$ , where  $\alpha(\mathbf{w}_{i,-i}, b_i) \equiv \sum_{\mathbf{x}_{-i}} 1[\mathbf{w}_{i,-i}^T \mathbf{x}_{-i} - b_i = 0]$ . This proves the lower bound  $\pi(\mathcal{G}) \geq \frac{1}{2}$ . Geometrically speaking,  $\alpha(\mathbf{w}_{i,-i}, b_i)$  is the number of vertices of the  $(n-1)$ -dimensional hypercube that are covered by the hyperplane with normal  $\mathbf{w}_{i,-i}$  and bias  $b_i$ . Recall that  $(\mathbf{w}_{i,-i}, b_i) \neq \mathbf{0}$ . If  $\mathbf{w}_{i,-i} = \mathbf{0}$  and  $b_i \neq 0$  then  $\alpha(\mathbf{w}_{i,-i}, b_i) = \sum_{\mathbf{x}_{-i}} 1[b_i = 0] = 0 \Rightarrow \pi(\mathcal{G}) = \frac{1}{2}$ . If  $\mathbf{w}_{i,-i} \neq \mathbf{0}$  then as noted in Aichholzer and Aurenhammer [1996] a hyperplane with  $n-2$  zeros on  $\mathbf{w}_{i,-i}$  (i.e. a  $(n-2)$ -parallel hyperplane) covers exactly half of the  $2^{n-1}$  vertices, the maximum possible. Therefore,  $\pi(\mathcal{G}) = \frac{1}{2} + \frac{1}{2^n} \alpha(\mathbf{w}_{i,-i}, b_i) \leq \frac{1}{2} + \frac{2^{n-2}}{2^n} = \frac{3}{4}$ .  $\square$

Next, we present our bound for the true proportion of equilibria of games in which all players are non-absolutely-indifferent.

**Theorem 21.** *If all players are non-absolutely-indifferent and if the rows of an influence game  $\mathcal{G} = (\mathbf{W}, \mathbf{b})$  are independent (but not necessarily identically distributed) random vectors, i.e. for every player  $i$ ,  $(\mathbf{w}_{i,-i}, b_i)$  is independently drawn from an arbitrary distribution  $\mathcal{P}_i$ , then the expected true proportion of equilibria is bounded as follows:*

$$(1/2)^n \leq \mathbb{E}_{\mathcal{P}_1, \dots, \mathcal{P}_n} [\pi(\mathcal{G})] \leq (3/4)^n \quad (21)$$

furthermore, the following high probability statement holds:

$$\mathbb{P}_{\mathcal{P}_1, \dots, \mathcal{P}_n} [\pi(\mathcal{G}) \leq \frac{(3/4)^n}{\delta}] \geq 1 - \delta \quad (22)$$

*Proof.* Let  $y_i \equiv 1[x_i(\mathbf{w}_{i,-i}^T \mathbf{x}_{-i} - b_i) \geq 0]$ ,  $\mathcal{P} \equiv \{\mathcal{P}_1, \dots, \mathcal{P}_n\}$  and  $\mathcal{U}$  the uniform distribution for  $\mathbf{x} \in \{-1, +1\}^n$ . By eq.(4),  $\mathbb{E}_{\mathcal{P}}[\pi(\mathcal{G})] = \mathbb{E}_{\mathcal{P}}[\frac{1}{2^n} \sum_{\mathbf{x}} \prod_i y_i] = \mathbb{E}_{\mathcal{P}}[\mathbb{E}_{\mathcal{U}}[\prod_i y_i]] = \mathbb{E}_{\mathcal{U}}[\mathbb{E}_{\mathcal{P}}[\prod_i y_i]]$ . Note that each  $y_i$  is independent since each  $(\mathbf{w}_{i,-i}, b_i)$  is independently distributed. Therefore,  $\mathbb{E}_{\mathcal{P}}[\pi(\mathcal{G})] = \mathbb{E}_{\mathcal{U}}[\prod_i \mathbb{E}_{\mathcal{P}_i}[y_i]]$ . Similarly each  $z_i \equiv \mathbb{E}_{\mathcal{P}_i}[y_i]$  is independent since each  $(\mathbf{w}_{i,-i}, b_i)$  is independently distributed. Therefore,  $\mathbb{E}_{\mathcal{P}}[\pi(\mathcal{G})] = \mathbb{E}_{\mathcal{U}}[\prod_i z_i] = \prod_i \mathbb{E}_{\mathcal{U}}[z_i] = \prod_i \mathbb{E}_{\mathcal{U}}[\mathbb{E}_{\mathcal{P}_i}[y_i]] = \prod_i \mathbb{E}_{\mathcal{P}_i}[\mathbb{E}_{\mathcal{U}}[y_i]]$ . Note that  $\mathbb{E}_{\mathcal{U}}[y_i]$  is the true proportion of equilibria of an influence game with non-absolutely-indifferent player  $i$  and absolutely-indifferent players  $\forall j \neq i$ , and therefore  $1/2 \leq \mathbb{E}_{\mathcal{U}}[y_i] \leq 3/4$  by Claim iii of Lemma 20. Finally, we have  $\mathbb{E}_{\mathcal{P}}[\pi(\mathcal{G})] \geq \prod_i \mathbb{E}_{\mathcal{P}_i}[1/2] = (1/2)^n$  and similarly  $\mathbb{E}_{\mathcal{P}}[\pi(\mathcal{G})] \leq \prod_i \mathbb{E}_{\mathcal{P}_i}[3/4] = (3/4)^n$ .

By Markov's inequality, given that  $\pi(\mathcal{G}) \geq 0$ , we have  $\mathbb{P}_{\mathcal{P}}[\pi(\mathcal{G}) \geq c] \leq \frac{\mathbb{E}_{\mathcal{P}}[\pi(\mathcal{G})]}{c} \leq \frac{(3/4)^n}{c}$ . For  $c = \frac{(3/4)^n}{\delta} \Rightarrow \mathbb{P}_{\mathcal{P}}[\pi(\mathcal{G}) \geq \frac{(3/4)^n}{\delta}] \leq \delta \Rightarrow \mathbb{P}_{\mathcal{P}}[\pi(\mathcal{G}) \leq \frac{(3/4)^n}{\delta}] \geq 1 - \delta$ .  $\square$

**Remark 22.** *Under the same assumptions of Theorem 21, it is possible to prove that with probability at least  $1 - \delta$  we have  $\pi(\mathcal{G}) \leq (3/4)^n + 3/8\sqrt{2\log \frac{1}{\delta}}$  by using Hoeffding’s lemma. We point out that such a bound is not better than the Markov’s bound derived above.*

## 8 Experimental Results

For learning influence games we used our convex loss methods: independent and simultaneous SVM and logistic regression. Additionally, we used the (super-exponential) exhaustive search method only for  $n \leq 4$ . As a baseline, we used the sigmoidal maximum likelihood (NP-hard) only for  $n \leq 15$  as well as the sigmoidal maximum empirical proportion of equilibria. Regarding the parameters  $\alpha$  and  $\beta$  our sigmoidal function in eq.(15), we found experimentally that  $\alpha = 0.1$  and  $\beta = 0.001$  achieved the best results.

We compare learning influence games to learning Ising models. For  $n \leq 15$  players, we perform exact  $\ell_1$ -regularized maximum likelihood estimation by using the FOBOS algorithm [Duchi and Singer, 2009a,b] and exact gradients of the log-likelihood of the Ising model. Since the computation of the exact gradient at each step is NP-hard, we used this method only for  $n \leq 15$ . For  $n > 15$  players, we use the Höfling-Tibshirani method [Höfling and Tibshirani, 2009], which uses a sequence of first-order approximations of the exact log-likelihood. We also used a two-step algorithm, by first learning the structure by  $\ell_1$ -regularized logistic regression [Wainwright et al., 2006] and then using the FOBOS algorithm [Duchi and Singer, 2009a,b] with belief propagation for gradient approximation. We did not find a statistically significant difference between the test log-likelihood of both algorithms and therefore we only report the latter.

Our experimental setup is as follows: after learning a model for different values of the regularization parameter  $\rho$  in a training set, we select the value of  $\rho$  that maximizes the log-likelihood in a validation set, and report statistics in a test set. For synthetic experiments, we report the Kullback-Leibler (KL) divergence, average precision (one minus the fraction of falsely included equilibria), average recall (one minus the fraction of falsely excluded equilibria) in order to measure the closeness of the recovered models to the ground truth. For real-world experiments, we report the log-likelihood. In both synthetic and real-world experiments, we report the number of equilibria and the empirical proportion of equilibria.

We first test the ability of the proposed methods to recover the ground truth structure from data. We use a small first synthetic model in order to compare with the (super-exponential) exhaustive search method. The ground truth model  $\mathcal{G}_g = (\mathbf{W}_g, \mathbf{b}_g)$  has  $n = 4$  players and 4 Nash equilibria (i.e.  $\pi(\mathcal{G}_g)=0.25$ ),  $\mathbf{W}_g$  was set according to Figure 3 (the weight of each edge was set to +1) and  $\mathbf{b}_g = \mathbf{0}$ . The mixture parameter of the ground truth  $q_g$  was set to 0.5,0.7,0.9. For each of 50 repetitions, we generated a training, a validation and a test set of 50 samples each. Figure 3 shows that our convex loss methods and sigmoidal maximum likelihood outperform (lower KL) exhaustive search, sigmoidal maximum empirical proportion of equilibria and Ising models. Note that the exhaustive search method which performs exact maximum likelihood suffers from over-fitting and consequently does not produce the lowest KL. From all convex loss methods, simultaneous logistic regression achieves the lowest KL. For all methods, the recovery of equilibria is perfect for  $q_g = 0.9$  (number of equilibria equal to the ground truth, equilibrium precision and recall equal to

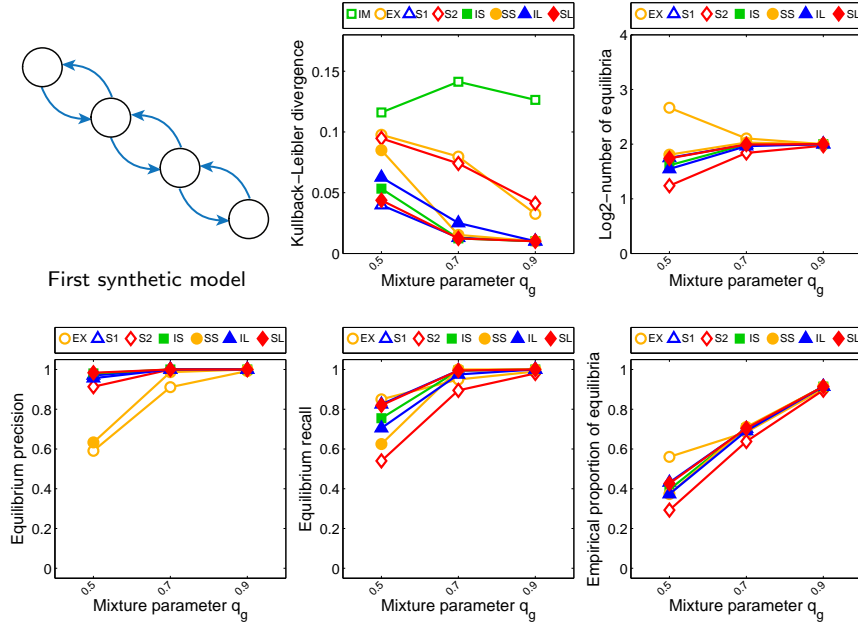


Figure 3: Closeness of the recovered models to the ground truth synthetic model for different mixture parameters  $q_g$ . Our convex loss methods (IS,SS: independent and simultaneous SVM, IL,SL: independent and simultaneous logistic regression) and sigmoidal maximum likelihood (S1) have lower KL than exhaustive search (EX), sigmoidal maximum empirical proportion of equilibria (S2) and Ising models (IM). For all methods, the recovery of equilibria is perfect for  $q_g = 0.9$  (number of equilibria equal to the ground truth, equilibrium precision and recall equal to 1) and the empirical proportion of equilibria resembles the mixture parameter of the ground truth  $q_g$ .

1). Additionally, the empirical proportion of equilibria resembles the mixture parameter of the ground truth  $q_g$ .

Next, we use a relatively larger second synthetic model with more complex interactions. We still keep the model small enough in order to compare with the (NP-hard) sigmoidal maximum likelihood method. The ground truth model  $\mathcal{G}_g = (\mathbf{W}_g, \mathbf{b}_g)$  has  $n = 9$  players and 16 Nash equilibria (i.e.  $\pi(\mathcal{G}_g) = 0.03125$ ),  $\mathbf{W}_g$  was set according to Figure 4 (the weight of each blue and red edge was set to  $+1$  and  $-1$  respectively) and  $\mathbf{b}_g = \mathbf{0}$ . The mixture parameter of the ground truth  $q_g$  was set to 0.5, 0.7, 0.9. For each of 50 repetitions, we generated a training, a validation and a test set of 50 samples each. Figure 4 shows that our convex loss methods outperform (lower KL) sigmoidal methods and Ising models. From all convex loss methods, simultaneous logistic regression achieves the lowest KL. For convex loss methods, the equilibrium recovery is better than the remaining methods (number of equilibria equal to the ground truth, higher equilibrium precision and recall). Additionally, the empirical proportion of equilibria resembles the mixture parameter of the ground truth  $q_g$ .

In the next experiment, we show that the performance of convex loss minimization improves as the number of samples increases. We used random graphs with slightly more variables and varying number of samples (10, 30, 100, 300). The ground truth model  $\mathcal{G}_g = (\mathbf{W}_g, \mathbf{b}_g)$  contains  $n = 20$  players. For each of 20 repetitions, we generate edges in the ground truth model  $\mathbf{W}_g$  with a required density (either 0.2, 0.5, 0.8). For simplicity, the

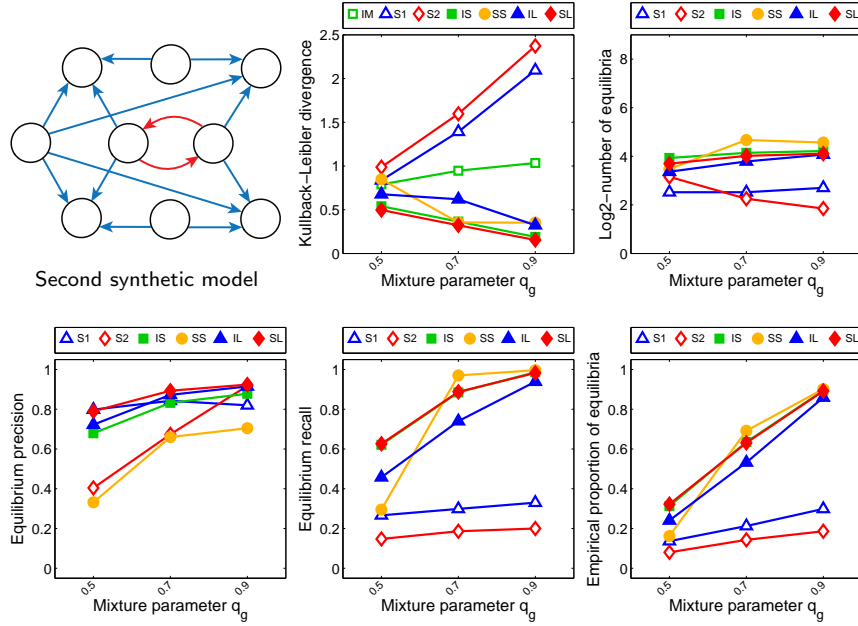


Figure 4: Closeness of the recovered models to the ground truth synthetic model for different mixture parameters  $q_g$ . Our convex loss methods (IS,SS: independent and simultaneous SVM, IL,SL: independent and simultaneous logistic regression) have lower KL than sigmoidal maximum likelihood (S1), sigmoidal maximum empirical proportion of equilibria (S2) and Ising models (IM). For convex loss methods, the equilibrium recovery is better than the remaining methods (number of equilibria equal to the ground truth, higher equilibrium precision and recall) and the empirical proportion of equilibria resembles the mixture parameter of the ground truth  $q_g$ .

weight of each edge is set to  $+1$  with probability  $P(+1)$  and to  $-1$  with probability  $1 - P(+1)$ . Hence, the Nash equilibria of the generated games does not depend on the magnitude of the weights, just on their sign. We set the bias  $\mathbf{b}_g = \mathbf{0}$  and the mixture parameter of the ground truth  $q_g = 0.7$ . We then generated a training and a validation set with the same number of samples. Figure 5 shows that our convex loss methods outperform (lower KL) sigmoidal maximum empirical proportion of equilibria and Ising models (except for the synthetic model with high true proportion of equilibria: density 0.8,  $P(+1) = 0$ ,  $NE > 1000$ ). The results are remarkably better when the number of equilibria in the ground truth model is small (e.g. for  $NE < 20$ ). From all convex loss methods, simultaneous logistic regression achieves the lowest KL.

In the next experiment, we evaluate two effects in our approximation methods. First, we evaluate the impact of removing the true proportion of equilibria from our objective function, i.e. the use of maximum empirical proportion of equilibria instead of maximum likelihood. Second, we evaluate the impact of using convex losses instead of a sigmoidal approximation of the 0/1 loss. We used random graphs with varying number of players and 50 samples. The ground truth model  $\mathcal{G}_g = (\mathbf{W}_g, \mathbf{b}_g)$  contains  $n = 4, 6, 8, 10, 12$  players. For each of 20 repetitions, we generate edges in the ground truth model  $\mathbf{W}_g$  with a required density (either 0.2, 0.5, 0.8). As in the previous experiment, the weight of each edge is set to  $+1$  with probability  $P(+1)$  and to  $-1$  with probability  $1 - P(+1)$ . We set the bias  $\mathbf{b}_g = \mathbf{0}$  and the mixture parameter of the ground truth  $q_g = 0.7$ . We then generated a training and a

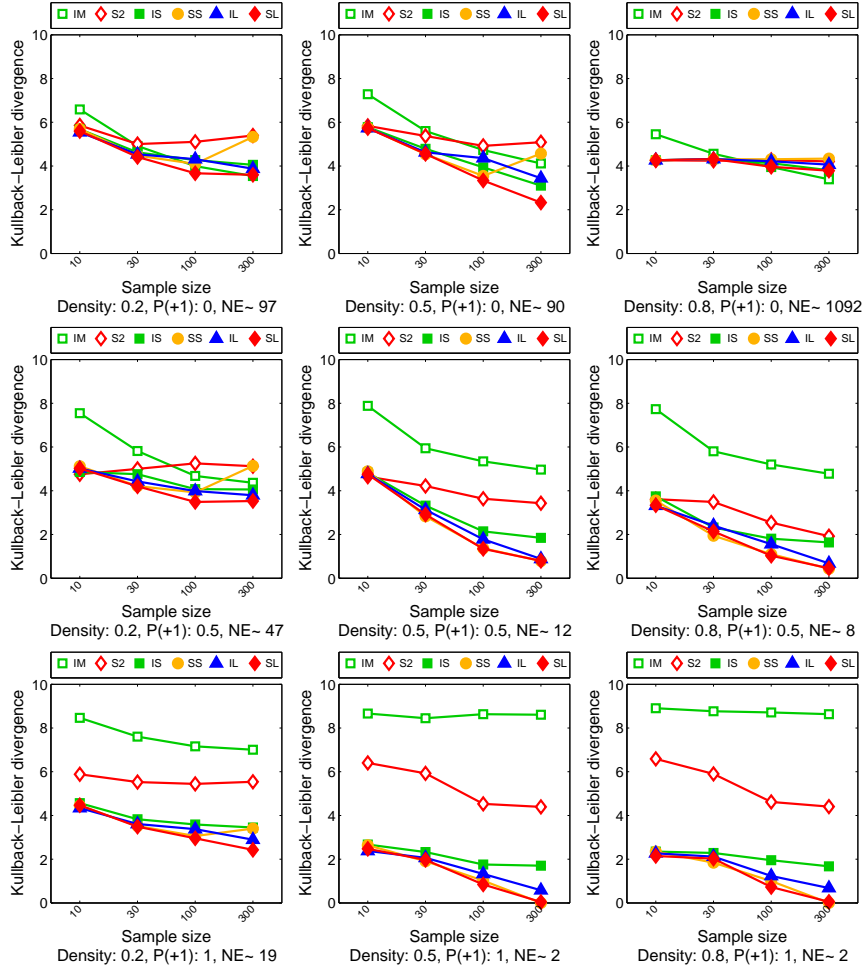


Figure 5: KL divergence between the recovered models and the ground truth for datasets of different number of samples. Each chart shows the density of the ground truth, probability  $P(+1)$  that an edge has weight  $+1$ , and average number of equilibria (NE). Our convex loss methods (IS,SS: independent and simultaneous SVM, IL,SL: independent and simultaneous logistic regression) have lower KL than sigmoidal maximum empirical proportion of equilibria (S2) and Ising models (IM). The results are remarkably better when the number of equilibria in the ground truth model is small (e.g. for  $NE < 20$ ).

validation set with the same number of samples. Figure 6 shows that in general, convex loss methods outperform (lower KL) sigmoidal maximum empirical proportion of equilibria, and the latter one outperforms sigmoidal maximum likelihood. A different effect is observed for mild (0.5) to high (0.8) density and  $P(+1) = 1$  in which the sigmoidal maximum likelihood obtains the lowest KL. In a closer inspection, we found that the ground truth games usually have only 2 equilibria:  $(+1, \dots, +1)$  and  $(-1, \dots, -1)$ , which seems to present a challenge for convex loss methods. It seems that for these specific cases, removing the true proportion of equilibria from the objective function negatively impacts the estimation process, but note that sigmoidal maximum likelihood is not computationally feasible for  $n > 15$ .

We used the U.S. congressional voting records in order to measure the generalization performance of convex loss minimization in a real-world dataset. The dataset is publicly

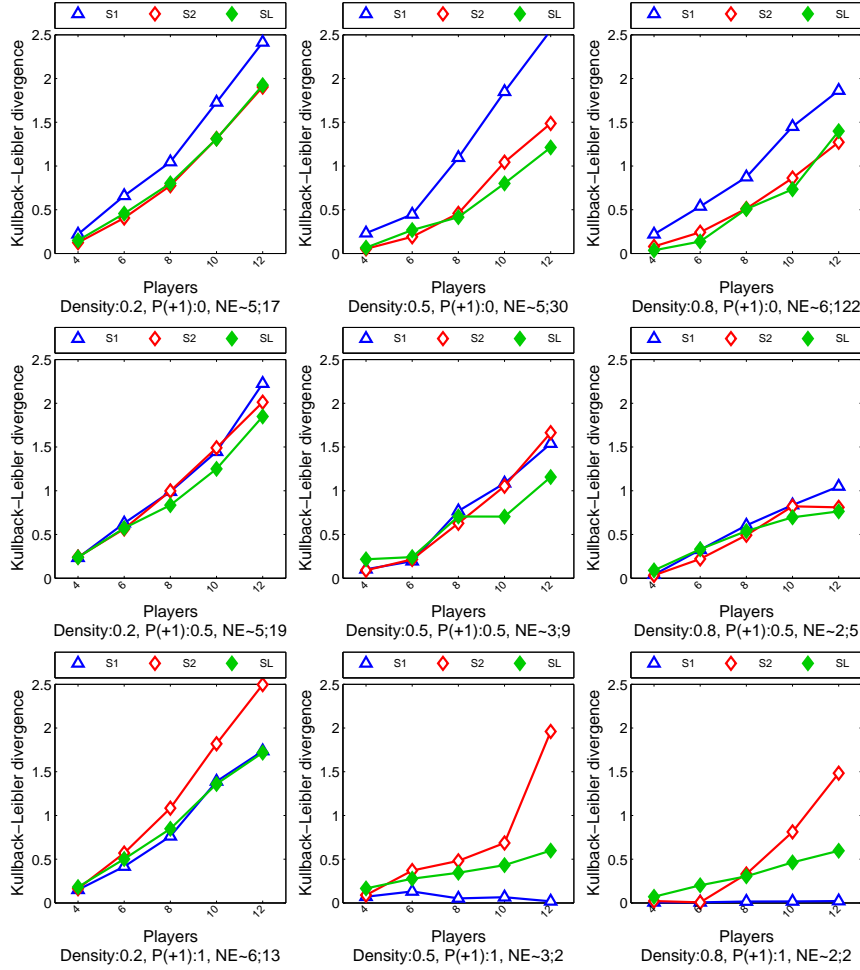


Figure 6: KL divergence between the recovered models and the ground truth for datasets of different number of players. Each chart shows the density of the ground truth, probability  $P(+1)$  that an edge has weight +1, and average number of equilibria (NE) for  $n = 2; n = 14$ . In general, simultaneous logistic regression (SL) has lower KL than sigmoidal maximum empirical proportion of equilibria (S2), and the latter one has lower KL than sigmoidal maximum likelihood (S1). Other convex losses behave the same as simultaneous logistic regression (omitted for clarity of presentation).

available at <http://www.senate.gov/>. We used the first session of the 104th congress (Jan 1995 to Jan 1996, 613 votes), the first session of the 107th congress (Jan 2001 to Dec 2001, 380 votes) and the second session of the 110th congress (Jan 2008 to Jan 2009, 215 votes). Following on other researchers who have experimented with this data set (e.g. Banerjee et al. [2008]), abstentions were replaced with negative votes. Since reporting the log-likelihood requires computing the number of equilibria (which is NP-hard), we selected only 20 senators by stratified random sampling. We randomly split the data into three parts. We performed six repetitions by making each third of the data take turns as training, validation and testing sets. Figure 7 shows that our convex loss methods outperform (higher log-likelihood) sigmoidal maximum empirical proportion of equilibria and Ising models. From all convex

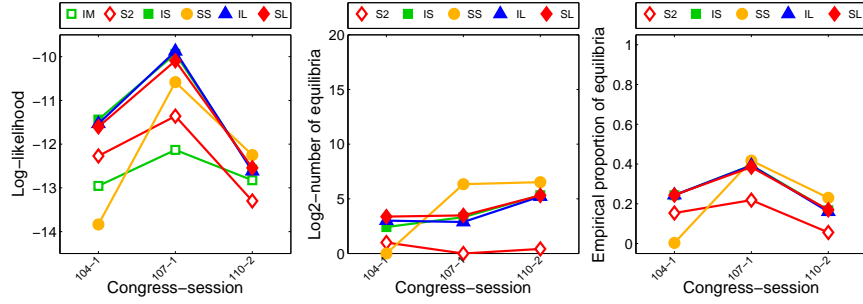


Figure 7: Statistics for games learnt from 20 senators from the first session of the 104th congress, first session of the 107th congress and second session of the 110th congress. The log-likelihood of our convex loss methods (IS,SS: independent and simultaneous SVM, IL,SL: independent and simultaneous logistic regression) is higher than sigmoidal maximum empirical proportion of equilibria (S2) and Ising models (IM). For all methods, the number of equilibria (and so the true proportion of equilibria) is low.

loss methods, simultaneous logistic regression achieves the lowest KL. For all methods, the number of equilibria (and so the true proportion of equilibria) is low.

We apply convex loss minimization to larger problems, by learning structures of games from all 100 senators. Figure 8 shows that simultaneous logistic regression produce structures that are sparser than its independent counterpart. The simultaneous method better elicits the bipartisan structure of the congress. We define the influence of player  $j$  to all other players as  $\sum_i |w_{ij}|$  after normalizing all weights, i.e. for each player  $i$  we divide  $(\mathbf{w}_{i,-i}, b_i)$  by  $\|\mathbf{w}_{i,-i}\|_1 + |b_i|$ . Note that Jeffords and Clinton are one of the 5 most directly-influential as well as 5 least directly-influenceable (high bias) senators, in the 107th and 110th congress respectively. McCain and Feingold are both in the list of 5 most directly-influential senators in the 104th and 107th congress. McCain appears again in the list of 5 least influenceable senators in the 110th congress.

We test the hypothesis that influence between senators of the same party are stronger than senators of different party. We learn structures of games from all 100 senators from the 101th congress to the 111th congress (Jan 1989 to Dec 2010). The number of votes casted for each session were average: 337, minimum: 215, maximum: 613. Figure 9 validates our hypothesis and more interestingly, it shows that influence between different parties is decreasing over time. Note that the influence from Obama to Republicans increased in the last sessions, while McCain’s influence to Republicans decreased.

## 9 Discussion

It is important to point out that our work is not in competition with the work in probabilistic graphical models, e.g. Ising models. Our goal is to learn the structure and parameters of games from data, and for this end, we propose a probabilistic model that is inspired by the concept of equilibrium in game theory. While we illustrate the benefit of our model in the U.S. congressional voting records, we believe that each model has its own benefits. If the practitioner “believes” that the data at hand is generated by a class of models, then the interpretation of the learnt model allows obtaining insight of the problem at hand. Note that none of the existing models (including ours) can be validated as the ground truth model that generated the real-world data, or as being more or less “realistic” with respect



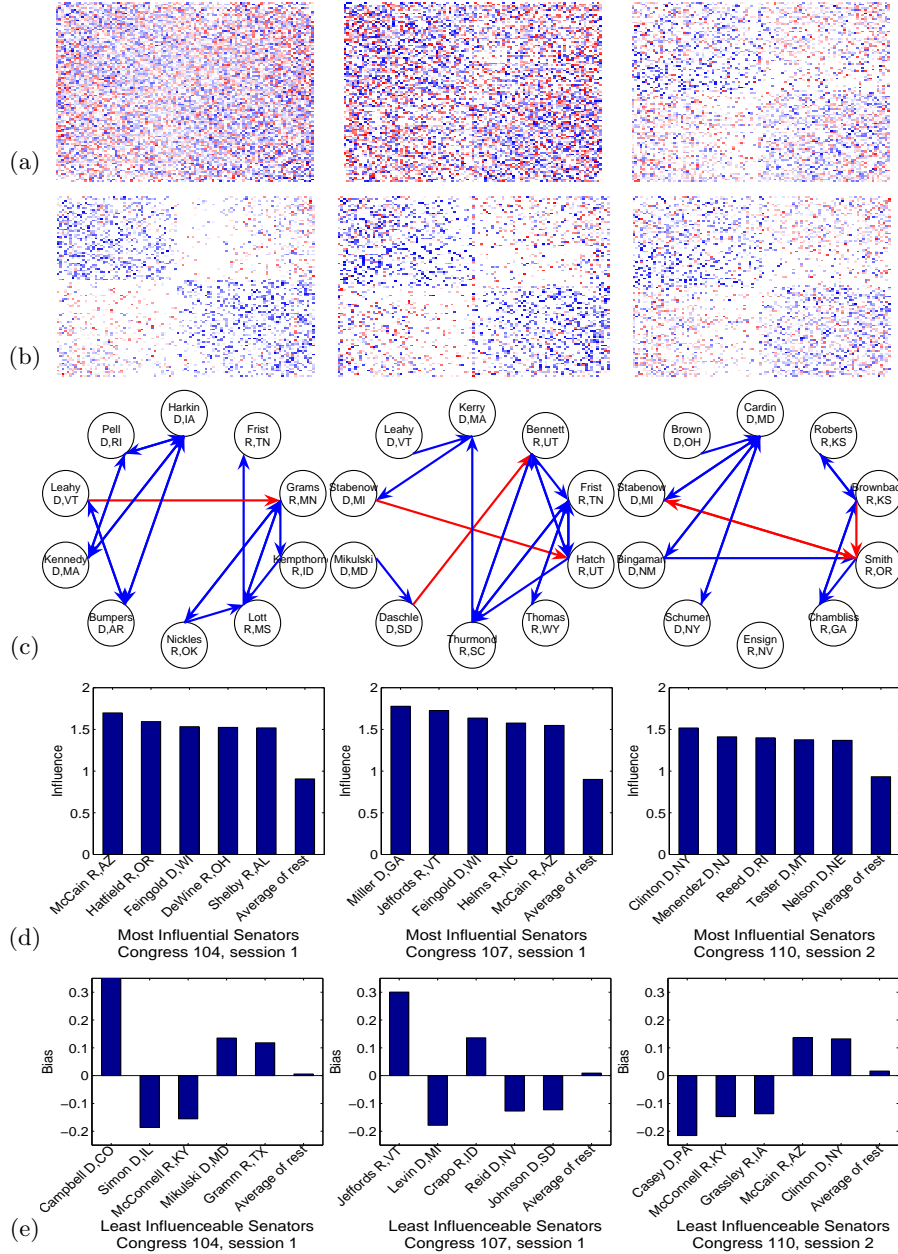


Figure 8: Matrix of influence weights for games learnt from all 100 senators, from the first session of the 104th congress (left), first session of the 107th congress (center) and second session of the 110th congress (right), by using our independent (a) and simultaneous (b) logistic regression methods. A row represents how every other senator influence the senator in such row. Positive influences are shown in blue, negative influences are shown in red. Democrats are shown in the top/left corner, while Republicans are shown in the bottom/right corner. Note that simultaneous method produce structures that are sparser than its independent counterpart. Partial view of the graph for simultaneous logistic regression (c). Most directly-influential (d) and least directly-influenceable (e) senators. Regularization parameter  $\rho = 0.0006$ .

to other model. While generalization in unseen data is a very important measurement, a model with better generalization is not the “ground truth model” of the real-world data at

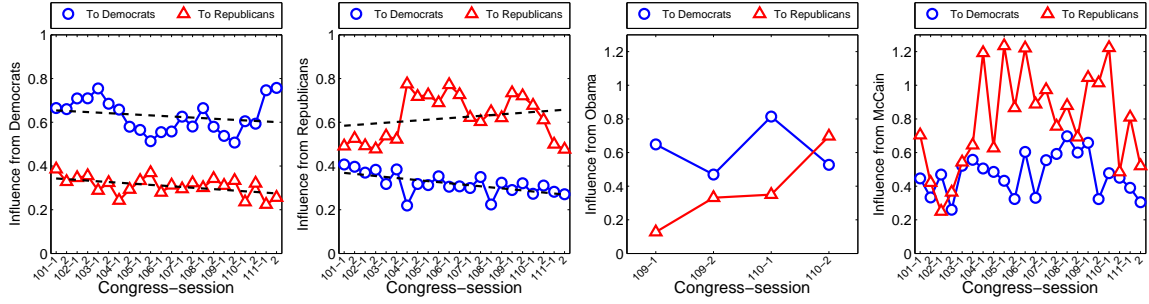


Figure 9: Direct influence between parties and influences from Obama and McCain. Games were learnt from all 100 senators from the 101th congress (Jan 1989) to the 111th congress (Dec 2010) by using our simultaneous logistic regression method. Direct influence between senators of the same party are stronger than senators of different party, which is also decreasing over time. In the last sessions, influence from Obama to Republicans increased, and influence from McCain to both parties decreased. Regularization parameter  $\rho = 0.0006$ .

hand. Finally, while our model is simple, it is well founded and we show that it is far from being computationally trivial. Therefore, we believe it has its own right to be analyzed.

The special class of graphical games considered here is related to the well-known *linear threshold model (LTM)* in sociology [Granovetter, 1978], recently very popular within the social network and theoretical computer science community [Kleinberg, 2007]. LTMs are usually studied as the basis for some kind of diffusion process. A typical problem is the identification of most influential individuals in a social network. An LTM is not in itself a game-theoretic model and, in fact, Granovetter himself argues against this view in the context of the setting and the type of questions in which he was most interested [Granovetter, 1978]. To the best of our knowledge, subsequent work on LTMs has not taken a strictly game-theoretic view either. Our model is also related to a particular model of *discrete choice with social interactions* in econometrics (see, e.g. Brock and Durlauf [2001]). The main difference is that we take a strictly non-cooperative game-theoretic approach within the classical “static”/one-shot game framework and do not use a *random utility model*. In addition, we do not make the assumption of *rational expectations*, which is equivalent to assuming that all players use exactly the same mixed strategy. As an aside note, regarding learning of information diffusion models over social networks, [Saito et al., 2010] considers a dynamic (continuous time) LTM that has only positive influence weights and a randomly generated threshold value.

There is still quite a bit of debate as to the appropriateness of game-theoretic equilibrium concepts to model individual human behavior in a social context. Camerer’s book on behavioral game theory [Camerer, 2003] addresses some of the issues. We point out that there is a broader view of behavioral data, beyond those generated by individual human behavior (e.g. institutions such as nations and industries, or engineered systems such as autonomous-response devices in residential or commercial properties that are programmed to control electricity usage based on user preferences). Our interpretation of Camerer’s position is not that Nash equilibria is universally a bad predictor but that it is not *consistently* the best, for reasons that are still not well understood. This point is best illustrated in Chapter 3, Figure 3.1 of Camerer [2003]. *Quantal response equilibria (QRE)* has been proposed as an alternative to Nash in the context of behavioral game theory. Models based

on QRE have been shown superior during *initial play* in some experimental settings, but most experimental work assume that the game’s payoff matrices are *known* and only the “precision parameter” is estimated, e.g. Wright and Leyton-Brown [2010]. Finally, most of the human-subject experiments in behavioral game theory involve only a handful of players, and the scalability of those results to games with *many* players is unclear.

In this work we considered pure-strategy Nash equilibria only. Note that the universality of mixed-strategy Nash equilibria does not diminish the importance of pure-strategy equilibria in game theory. Indeed, a debate still exist within the game theory community as to the justification for randomization, specially in human contexts. We decided to ignore mixed-strategies due to the significant added complexity. Note that we learn exclusively from observed joint-actions, and therefore we cannot assume knowledge of the internal mixed-strategies of players. We could generalize our model to allow for mixed-strategies by defining a process in which a joint mixed strategy  $\mathcal{P}$  from the set of mixed-strategy Nash equilibrium (or its complement) is drawn according to some distribution, then a (pure-strategy) realization  $\mathbf{x}$  is drawn from  $\mathcal{P}$  that would correspond to the observed joint-actions.

In this paper we considered a “global” noise process, which is governed with a probability  $q$  of selecting an equilibrium. Potentially better and more natural “local” noise processes are possible, at the expense of producing a significantly more complex generative model than the one considered in this paper. For instance, we could use a noise process that is formed of many independent, individual noise processes, one for each player. As an example, consider a the generative model in which we first select an equilibrium  $\mathbf{x}$  of the game and then each player  $i$ , independently, acts according to  $x_i$  with probability  $q_i$  and switches its action with probability  $1 - q_i$ . The problem with such a model is that it leads to a significantly more complex expression for the generative model and thus likelihood functions. This is in contrast to the simplicity afforded us by the generative model with a more global noise process defined above.

## 10 Concluding Remarks

There are several ways of extending this research. Different upper bounds for the 0/1 loss (e.g. exponential, smooth hinge) as well as  $\ell_2$ -norm regularizers need to be analyzed. Learning structures in settings in which players can take more than two possible actions or follow non-linear (e.g. kernelized) strategies, need to be investigated. More sophisticated noise processes as well as mixed-strategy Nash equilibria need to be considered and studied. Finally, topic-specific and time-varying versions of our model would elicit differences in preferences and trends.

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## A Negative Results

### A.1 Counting the Number of Equilibria is NP-hard

Here we provide a proof that establishes NP-hardness of counting the number of Nash equilibria, and thus also of evaluating the log-likelihood function for our generative model. A #P-hardness proof was originally provided by Irfan and Ortiz [2011], here we present a related proof for completeness. The reduction is from the *set partition problem* for a specific instance of a single *non-absolutely-indifferent* player.

Recall the *set partition problem*: given a multiset of  $n$  positive numbers  $\{a_1, \dots, a_n\}$ ,  $\text{SetPartition}(\mathbf{a})$  answers “yes” if and only if it is possible to partition the numbers into two

disjoint subsets  $\mathcal{S}_1$  and  $\mathcal{S}_2$  such that  $\mathcal{S}_1 \cap \mathcal{S}_2 = \emptyset$ ,  $\mathcal{S}_1 \cup \mathcal{S}_2 = \{1, \dots, n\}$  and  $\sum_{i \in \mathcal{S}_1} a_i - \sum_{i \in \mathcal{S}_2} a_i = 0$ ; otherwise it answers “no”. The set partition problem is equivalent to the *subset sum problem*, in which given a set of positive numbers  $\{a_1, \dots, a_n\}$  and a target sum  $c > 0$ ,  $\text{SubSetSum}(\mathbf{a}, c)$  answers “yes” if and only if there is a subset  $\mathcal{S} \subset \{1, \dots, n\}$  such that  $\sum_{i \in \mathcal{S}} a_i = c$ ; otherwise it answers “no”. The equivalence between set partition and subset sum follows from  $\text{SetPartition}(\mathbf{a}) = \text{SubSetSum}(\mathbf{a}, \frac{1}{2} \sum_i a_i)$ .

For clarity of exposition, we drop the subindices in the following lemma. Let  $\mathbf{w} \equiv \mathbf{w}_{i,-i} \in \mathbb{R}^{n-1}$  and  $b \equiv b_i \in \mathbb{R}$ .

**Lemma 23.** *The problem of counting Nash equilibria considered in Claim ii of Lemma 20 reduces to the set partition problem. More specifically, given  $(\forall i) w_i > 0, b = 0$ , answering whether  $\sum_{\mathbf{x}} 1[\mathbf{w}^T \mathbf{x} - b = 0] > 0$  is equivalent to answering  $\text{SetPartition}(\mathbf{w})$ .*

*Proof.* Let  $\mathcal{S}_1(\mathbf{x}) = \{i | x_i = +1\}$  and  $\mathcal{S}_2(\mathbf{x}) = \{i | x_i = -1\}$ . We can rewrite  $\sum_{\mathbf{x}} 1[\mathbf{w}^T \mathbf{x} - b = 0]$  as a sum of *set partition* conditions, i.e.  $\sum_{\mathbf{x}} 1[\sum_{i \in \mathcal{S}_1(\mathbf{x})} w_i - \sum_{i \in \mathcal{S}_2(\mathbf{x})} w_i = 0]$ . Therefore, if no tuple  $\mathbf{x}$  fulfills the condition, the sum is zero and  $\text{SetPartition}(\mathbf{w})$  answers “no”. On the other hand, if at least one tuple  $\mathbf{x}$  fulfills the condition, the sum is greater than zero and  $\text{SetPartition}(\mathbf{w})$  answers “yes”.  $\square$

## A.2 Computing the Pseudo-Likelihood is NP-hard

We show that evaluating the pseudo-likelihood function for our generative model is NP-hard. First, consider a non-trivial influence game  $\mathcal{G}$  in which eq.(3) simplifies to  $p_{(\mathcal{G}, q)}(\mathbf{x}) = q \frac{1[\mathbf{x} \in \mathcal{NE}(\mathcal{G})]}{|\mathcal{NE}(\mathcal{G})|} + (1 - q) \frac{1[\mathbf{x} \notin \mathcal{NE}(\mathcal{G})]}{2^n - |\mathcal{NE}(\mathcal{G})|}$ . Furthermore, assume the game  $\mathcal{G} = (\mathbf{W}, \mathbf{b})$  has a single *non-absolutely-indifferent* player  $i$  and *absolutely-indifferent* players  $\forall j \neq i$ . Let  $f_i(\mathbf{x}_{-i}) \equiv \mathbf{w}_{i,-i}^T \mathbf{x}_{-i} - b_i$ . By Claim i of Lemma 20, we have  $1[\mathbf{x} \in \mathcal{NE}(\mathcal{G})] = 1[x_i f_i(\mathbf{x}_{-i}) \geq 0]$  and therefore  $p_{(\mathcal{G}, q)}(\mathbf{x}) = q \frac{1[x_i f_i(\mathbf{x}_{-i}) \geq 0]}{|\mathcal{NE}(\mathcal{G})|} + (1 - q) \frac{1 - 1[x_i f_i(\mathbf{x}_{-i}) \geq 0]}{2^n - |\mathcal{NE}(\mathcal{G})|}$ . Finally, by Lemma 23 computing  $|\mathcal{NE}(\mathcal{G})|$  is NP-hard even for this specific instance of a single *non-absolutely-indifferent* player.

## A.3 Counting the Number of Equilibria is not (Lipschitz) Continuous

We show that small changes in the parameters  $\mathcal{G} = (\mathbf{W}, \mathbf{b})$  can produce big changes in  $|\mathcal{NE}(\mathcal{G})|$ . For instance, consider two games  $\mathcal{G}_k = (\mathbf{W}_k, \mathbf{b}_k)$ , where  $\mathbf{W}_1 = \mathbf{0}, \mathbf{b}_1 = \mathbf{0}, |\mathcal{NE}(\mathcal{G}_1)| = 2^n$  and  $\mathbf{W}_2 = \varepsilon(\mathbf{1}\mathbf{1}^T - \mathbf{I}), \mathbf{b}_2 = \mathbf{0}, |\mathcal{NE}(\mathcal{G}_2)| = 2$  for  $\varepsilon > 0$ . For  $\varepsilon \rightarrow 0$ , any  $\ell_p$ -norm  $\|\mathbf{W}_1 - \mathbf{W}_2\|_p \rightarrow 0$  but  $|\mathcal{NE}(\mathcal{G}_1)| - |\mathcal{NE}(\mathcal{G}_2)| = 2^n - 2$  remains constant.

## A.4 The Log-Partition Function of an Ising Model is a Trivial Bound for Counting the Number of Equilibria

Let  $f_i(\mathbf{x}_{-i}) \equiv \mathbf{w}_{i,-i}^T \mathbf{x}_{-i} - b_i$ ,  $|\mathcal{NE}(\mathcal{G})| = \sum_{\mathbf{x}} \prod_i 1[x_i f_i(\mathbf{x}_{-i}) \geq 0] \leq \sum_{\mathbf{x}} \prod_i e^{x_i f_i(\mathbf{x}_{-i})} = \sum_{\mathbf{x}} e^{\mathbf{x}^T \mathbf{W} \mathbf{x} - \mathbf{b}^T \mathbf{x}} = \mathcal{Z}(\frac{1}{2}(\mathbf{W} + \mathbf{W}^T), \mathbf{b})$ , where  $\mathcal{Z}$  denotes the partition function of an Ising model. Given convexity of  $\mathcal{Z}$  [Koller and Friedman, 2009] and that the gradient vanishes at  $\mathbf{W} = \mathbf{0}, \mathbf{b} = \mathbf{0}$ , we know that  $\mathcal{Z}(\frac{1}{2}(\mathbf{W} + \mathbf{W}^T), \mathbf{b}) \geq 2^n$ , which is the maximum  $|\mathcal{NE}(\mathcal{G})|$ .

## B Simultaneous Logistic Loss

Given that any loss  $\ell(z)$  is a decreasing function, the following identity holds  $\max_i \ell(z_i) = \ell(\min_i z_i)$ . Hence, we can either upper-bound the max function by the logsumexp function or lower-bound the min function by a negative logsumexp. We chose the latter option for the logistic loss for the following reasons: Claim i of the following technical lemma shows that lower-bounding min generates a loss that is strictly less than upper-bounding max. Claim ii shows that lower-bounding min generates a loss that is strictly less than independently penalizing each player. Claim iii shows that there are some cases in which upper-bounding max generates a loss that is strictly greater than independently penalizing each player.

**Lemma 24.** *For the logistic loss  $\ell(z) = \log(1+e^{-z})$  and a set of  $n > 1$  numbers  $\{z_1, \dots, z_n\}$ :*

- i.  $(\forall z_1, \dots, z_n) \max_i \ell(z_i) \leq \ell(-\log \sum_i e^{-z_i}) < \log \sum_i e^{\ell(z_i)} \leq \max_i \ell(z_i) + \log n$
  - ii.  $(\forall z_1, \dots, z_n) \ell(-\log \sum_i e^{-z_i}) < \sum_i \ell(z_i)$
  - iii.  $(\exists z_1, \dots, z_n) \log \sum_i e^{\ell(z_i)} > \sum_i \ell(z_i)$
- (23)

*Proof.* Given a set of numbers  $\{a_1, \dots, a_n\}$ , the max function is bounded by the logsumexp function by  $\max_i a_i \leq \log \sum_i e^{a_i} \leq \max_i a_i + \log n$  [Boyd and Vandenberghe, 2006]. Equivalently, the min function is bounded by  $\min_i a_i - \log n \leq -\log \sum_i e^{-a_i} \leq \min_i a_i$ .

These identities allow us to prove two inequalities in Claim i, i.e.  $\max_i \ell(z_i) = \ell(\min_i z_i) \leq \ell(-\log \sum_i e^{-z_i})$  and  $\log \sum_i e^{\ell(z_i)} \leq \max_i \ell(z_i) + \log n$ . To prove the remaining inequality  $\ell(-\log \sum_i e^{-z_i}) < \log \sum_i e^{\ell(z_i)}$ , note that for the logistic loss  $\ell(-\log \sum_i e^{-z_i}) = \log(1 + \sum_i e^{-z_i})$  and  $\log \sum_i e^{\ell(z_i)} = \log(n + \sum_i e^{-z_i})$ . Since  $n > 1$ , strict inequality holds.

To prove Claim ii, we need to show that  $\ell(-\log \sum_i e^{-z_i}) = \log(1 + \sum_i e^{-z_i}) < \sum_i \ell(z_i) = \sum_i \log(1 + e^{-z_i})$ . This is equivalent to  $1 + \sum_i e^{-z_i} < \prod_i (1 + e^{-z_i}) = \sum_{\mathbf{c} \in \{0,1\}^n} e^{-\mathbf{c}^T \mathbf{z}} = 1 + \sum_i e^{-z_i} + \sum_{\mathbf{c} \in \{0,1\}^n, \mathbf{1}^T \mathbf{c} > 1} e^{-\mathbf{c}^T \mathbf{z}}$ . Finally, we have  $\sum_{\mathbf{c} \in \{0,1\}^n, \mathbf{1}^T \mathbf{c} > 1} e^{-\mathbf{c}^T \mathbf{z}} > 0$  because the exponential function is strictly positive.

To prove Claim iii, it suffices to find set of numbers  $\{z_1, \dots, z_n\}$  for which  $\log \sum_i e^{\ell(z_i)} = \log(n + \sum_i e^{-z_i}) > \sum_i \ell(z_i) = \sum_i \log(1 + e^{-z_i})$ . This is equivalent to  $n + \sum_i e^{-z_i} > \prod_i (1 + e^{-z_i})$ . By setting  $(\forall i) z_i = \log n$ , we reduce the claim we want to prove to  $n + 1 > (1 + \frac{1}{n})^n$ . Strict inequality holds for  $n > 1$ . Furthermore, note that  $\lim_{n \rightarrow +\infty} (1 + \frac{1}{n})^n = e$ .  $\square$